Scalar Curvature Rigidity Theorems for the Upper Hemisphere

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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Abstract
(Mathematics)

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Abstract

In this dissertation we study scalar curvature rigidity phenomena for the upper hemisphere, and subsets thereof. In particular, we are interested in Min–Oo's conjecture that there exist no metrics on the upper hemisphere having scalar curvature greater or equal to that of the standard spherical metric, while satisfying certain natural geometric boundary conditions.

While the conjecture as originally stated has recently been disproved, there are still many interesting modifications to consider. For instance, it has been shown that Min–Oo's rigidity conjecture holds on sufficiently small geodesic balls contained in the upper hemisphere, for metrics sufficiently close to the spherical metric. We show that this local rigidity phenomena can be extended to a larger class of domains in the hemisphere, in particular finding that it holds on larger geodesic balls, and on certain domains other than geodesic balls (which necessarily have more complicated boundary geometry). We discuss a possible method for finding the largest possible domain on which the local rigidity theorem is true, and give a Morse–theoretic interpretation of the problem.

Another interesting open question is whether or not such a rigidity statement holds for metrics that are not close to the spherical metric. We provide positive evidence for a conjecture that a scalar curvature rigidity theorem can be proved for metrics on sufficiently small geodesic balls in the hemisphere, provided certain additional geometric constraints are satisfied.

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1

Introduction

The strongest local invariant of a Riemannian metric is the Riemann curvature tensor, followed by the Ricci curvature and then the scalar curvature. While the scalar curvature is the weakest of these invariants, it is, from an algebraic standpoint, the simplest measurement of curvature, being a scalar function as opposed to a higher-rank tensorial object. For any point p in a Riemannian manifold (M^n, g) one has the following expansion for the volume of a geodesic ball centered at p:

$$\frac{\mathrm{Vol}(B_r(p))}{\omega_n r^n} = 1 - \frac{R(p)}{6(n-2)} r^2 + \mathcal{O}(r^4) \tag{1.1}$$

where $R(p)$ is the scalar curvature at p, and ω_n is the volume of the unit ball in $\mathbb{R}^n$. Thus, on an infinitesimal level, the scalar curvature controls the volume of geodesic balls. Globally, the following result has been shown.

Theorem 1. *(Bray (1997)) Let $\bar{g}$ denote a constant curvature metric on $\mathbb{S}^3$, with $\overline{\mathrm{Ric}} = \lambda \bar{g}$ and $\bar{R} = 3\lambda$. There exists a constant $\epsilon_0 \in (0,1)$ such that any complete, smooth Riemannian manifold (M^3, g) with $R_g \geq 3\lambda$ and $\mathrm{Ric}_g \geq \epsilon_0 \lambda g$ has volume*

$$\mathrm{Vol}(M, g) \leq \mathrm{Vol}(S^3, \bar{g}).$$

This theorem shows that a lower bound on scalar curvature results in an upper bound on volume, provided the Ricci curvature is sufficiently positive. In the absence of a positive lower bound on the Ricci curvature, there exist counterexamples showing the result to be false. This demonstrates that the scalar curvature on its own is not sufficiently strong to yield a global volume comparison.

Conversely, it is known in special cases that the volume function $r \mapsto \mathrm{Vol}(B_r(p))$ controls the geometry of (M, g) near p.

Theorem 2. *(Gray and Vanhecke (1979)) Suppose* $\mathrm{Vol}(B_r(p)) = \omega_n r^n$ *for all* $p \in M$ *and all sufficiently small* $r > 0$*. If any of the following conditions are satisfied*

- $\dim M \leq 3$,
- Ric_g *is nonnegative or nonpositive,*
- (M, g) *is conformally flat,*
- M *is a compact, oriented 4-manifold whose Euler characteristic and signature satisfy* $\chi(M) \geq -\frac{2}{3}|\tau(M)|$,
- M *is a product of spheres,*

then (M, g) *is flat.*

The theorem is proved by considering higher-order terms in the power series expansion (1.1), the coefficients of which are expressible as universal polynomials in the curvature tensor and its derivatives. To demonstrate that this is nontrivial, we mention that there exists a non-flat Riemannian 4-manifold with

$$\mathrm{Vol}(B_r(p)) = \omega_n r^n \left(1 + \mathcal{O}(r^6)\right),$$

and a manifold of dimension 734 with

$$\mathrm{Vol}(B_r(p)) = \omega_n r^n \left(1 + \mathcal{O}(r^8)\right),$$

for all p.

1.1 Scalar curvature rigidity phenomena

It is an immediate consequence of the celebrated positive mass theorem (Schoen and Yau (1979), Witten (1981), Parker and Taubes (1982)) that any complete Riemannian manifold (M^n, g) that is isometric to Euclidean space outside a compact set, and has nonnegative scalar curvature, is in fact globally isometric to Euclidean space. (If $n > 7$ it is necessary to assume that M is a spin manifold.) This implies that any compactly supported, non-isometric deformation g of the standard Euclidean metric must have $R_g < 0$ somewhere. Equivalently, any metric on the torus $\mathbb{T}^n$ must have negative scalar curvature somewhere, unless is it flat. A local version of this result appeared in Fischer and Marsden (1975), preceding the proof of the positive mass theorem.

Theorem 3. *(Fischer and Marsden (1975)) Suppose (M^n, g_0) is compact and flat. For any real number $p > n$, there exists a neighborhood U of g_0, in the $W^{2,p}$ topology, such that $g \in U$ and $R_g \geq 0$ implies $g = \varphi^* g_0$ for some diffeomorphism φ of M.*

Given a Riemannian metric g and a function S that is close to R_g in an appropriate sense, it is in general possible to perturb g to a nearby metric $\tilde{g}$ that has scalar curvature $R_{\tilde{g}} = S$. To make this more precise, we let L_g denote the linearization of the scalar curvature operator, $g \mapsto R_g$. The L^2-formal adjoint is given by

$$L_g^* u = \operatorname{Hess} u - (\Delta u) g - u \operatorname{Ric}$$

for any function u. We say a metric g is *static* if $\ker L_g^*$ is nontrivial. We then have the following:

Theorem 4. *(Corvino (2000)) Let Ω be a smooth domain compactly contained in a smooth Riemannian manifold (M, g). Suppose $L_g^* : W^{2,2}_{loc}(\Omega) \to L^2_{loc}(\Omega)$ is injective. Then for any smooth function S with* $\operatorname{supp}(S - R_g) \Subset \Omega$ *and $\|S - R_g\|_{C^\alpha}$ sufficiently* *small, there exists a smooth Riemannian metric $\tilde{g}$ on M such that $R_{\tilde{g}} = S$ in Ω and $\tilde{g} \equiv g$ outside Ω.*

A similar result was shown in Fischer and Marsden (1975) for compact manifolds; the contribution of Corvino is to localize the deformation to a specified compact domain. Viewing Theorem 3 in light of this result, we see that it is not possible to pointwise increase the scalar curvature of $\mathbb{T}^n$ because g_0 is static, with $\ker L^*_{g_0} = \mathbb{R}$ comprising the constant functions. However, the condition $\ker L^*_g = \{0\}$ is satisfied for generic metrics, as demonstrated by the following theorem (*cf.* Besse (1987)).

Theorem 5. *(Bourguignon (1975)) If $\ker L^*_g \neq \{0\}$ then either (M^n, g) is Ricci-flat and $\ker L^*_g = \mathbb{R}$, or the scalar curvature is a strictly positive constant and $-R_g/(n-1)$ is an eigenvalue of the Laplacian.*

This implies that the set of non-static metrics on M is open and dense in the $W^{2,p}$ topology for any $p > n$.

While the above results present obstructions to increasing the scalar curvature of a Riemannian manifold, the following remarkable result shows that there is no corresponding obstruction to pointwise *decreasing* the scalar curvature of a metric.

Theorem 6. *(Lohkamp (1999)) Let U be an open subset in a smooth Riemannian manifold (M^n, g), with $n \geq 3$. Suppose $f \in C^\infty(M)$ satisfies $f < R_g$ in U and $f \equiv R_g$ in $M \setminus U$. Then for each $\epsilon > 0$ there exists a smooth metric g_ϵ on M with $g_\epsilon \equiv g$ in $M \setminus U_\epsilon$ and*

$$f - \epsilon \leq R_{g_\epsilon} \leq f$$

in U_ϵ, where U_ϵ denotes the ϵ-neighborhood of U computed with respect to g.

1.2 Min–Oo's conjecture

In the above discussion of the positive mass theorem, it was seen that the behavior of a Riemannian metric at infinity imposes restrictions on the geometry of the interior, and in particular the possible values of the scalar curvature. The following result extends this idea to compact manifolds with boundary, where the interior geometry is now controlled by the geometry at the boundary.

Theorem 7. *(Miao (2002)) Let g be a metric on the closed unit ball $B \subset \mathbb{R}^n$ such that ∂B is isometric to $\mathbb{S}^{n-1}$, and has mean curvature $H_g > n-1$. Then $R_g(p) < 0$ for some point $p \in B$.*

In fact it suffices to have the mean curvature satisfy $H \geq n-1$, with strict inequality as a point. In the case $n = 3$ this leads to a rigidity statement for the flat metric on the unit ball.

Theorem 8. *(Miao (2002)) Let g be a metric on the closed unit ball $B \subset \mathbb{R}^3$ such that ∂B is isometric to $\mathbb{S}^2$, and has mean curvature $H_g \geq 2$. If $R_g \geq 0$ in B, then g is isometric to the standard Euclidean metric, g_0.*

This was later generalized by Shi and Tam to a rigidity statement for arbitrary convex domains in $\mathbb{R}^n$. We recall that a domain in $\mathbb{R}^n$ is said to be convex if the second fundamental form of its boundary (computed with respect to the Euclidean metric) is nonnegative, and strictly convex if the second fundamental form is positive definite.

Theorem 9. *(Shi and Tam (2002)) Let $\Omega \subset \mathbb{R}^n$ be a smooth, strictly convex domain, with boundary Σ. Suppose g is a Riemannian metric on Ω such that:*

- *$R_g \geq 0$ in Ω;*
- *(Σ, g) is isometric to (Σ, g_0);*
- *$H_g > 0$ on Σ.*

Then

$$\int_\Sigma (H_0 - H_g)\, d\mu_g \geq 0,$$

with equality if and only if g is isometric to g_0.

Based on similar considerations, Min–Oo made the following rigidity conjecture for the upper hemisphere.

Conjecture 10. *(Min-Oo, 1995) Let (M^n, g) be a compact Riemannian manifold with boundary. Suppose that $R_g \geq n(n-1)$, and that ∂M is totally geodesic and isometric to $\mathbb{S}^{n-1}$. Then M is isometric to $\mathbb{S}^n_+$ with the standard spherical metric, $\bar{g}$.*

This result is now known to be false, due to a recent counterexample of Brendle et al. (2010). However, there are certain restricted cases in which Min–Oo's original conjecture, or a suitable variation thereof, can be verified. We review these results in the following section.

1.3 A survey of known results

We begin by discussing the case $n = 2$, where Min–Oo's conjecture is true as originally stated. In the case we have the following theorem of Toponogov on the length of closed geodesics on convex surfaces.

Theorem 11. *(Toponogov (1959)) Let Σ be a closed surface with Gaussian curvature $K \geq 1$. Then any simple, closed geodesic γ in Σ has length $L(\gamma) \leq 2\pi$. If there exists a simple, closed geodesic of length 2π, then Σ is isometric to $\mathbb{S}^2$ with the standard metric.*

Although the result is stated for closed surfaces, it is immediate from the proof (*cf.* Klingenberg (1995)) that the theorem is true when Σ is a compact surface with totally geodesic boundary. Thus any compact surface Σ with $K \geq 1$, such that $\partial\Sigma$ is totally geodesic and has length 2π, is necessarily isometric to the upper hemisphere $\mathbb{S}^2_+$. This establishes Min–Oo's conjecture in dimension two. An alternate proof has been given recently, making use of the Gauss–Bonnet formula and the uniformization theorem.

Theorem 12. *(Hang and Wang (2009)) Let* (Σ, g) *be a compact surface with boundary, having Gaussian curvature* $K \geq 1$*. Suppose the geodesic curvature* k *of the boundary* γ *satisfies* $k \geq c \geq 0$*. Then* $L(\gamma) \leq 2\pi/\sqrt{1+c^2}$*. Moreover, equality holds if and only if* (Σ, g) *is isometric to a disc of radius* $\cot^{-1}(c)$ *in* $\mathbb{S}^2$*.*

As mentioned above, the proof utilizes conformal arguments that are only available in two dimensions. However, similar ideas can be used in higher dimensions to prove a version of Min–Oo's conjecture for metrics that are conformally equivalent to the standard spherical metric, $\bar{g}$.

Theorem 13. *(Hang and Wang (2009)) Assume* $\Omega \subset \mathbb{S}^n$ *is a smooth domain with metric* $g = u^{\frac{4}{n-2}}\bar{g}$*, such that* $R_g \geq n(n-1)$ *and* $u|_{\partial\Omega} = 1$ *Then* $u \geq 1$ *and* $H_g \leq H_{\bar{g}}$*. Moreover, if* $H_g = H_{\bar{g}}$ *at any point on* $\partial\Omega$*, or* $u = 1$ *at any point in the interior of* Ω*, then* $u \equiv 1$*.*

Additionally, it can be shown that the conjecture is false for domains strictly larger than the hemisphere, even when one restricts attention to the conformal class of $\bar{g}$. We let $B(N, r)$ denote the geodesic ball of radius r centered at the north pole, so that $B\left(N, \frac{\pi}{2}\right) = \mathbb{S}^n_+$.

Theorem 14. *(Hang and Wang (2006)) For any* $r \in (\frac{\pi}{2}, \pi)$ *there is a smooth metric* $g = e^{2\phi}\bar{g}$ *on* $\mathbb{S}^n$ *with the following properties:*

- $R_g \geq n(n-1)$*;*
- $\operatorname{supp}\phi \subset B(N, r)$*;*
- $\phi \neq 0$*.*

The next positive result in the direction of Min–Oo's conjecture is

Theorem 15. *(Eichmair (2009)) Let* (M^3, g) *be a compact, orientable Riemannian manifold with scalar curvature* $R_g \geq 6$*, Ricci curvature* $\mathrm{Ric}_g > 0$ *and totally geodesic* *boundary ∂M. If ∂M has area $|\partial M| \geq 4\pi$ and is an isoperimetric surface for the doubled manifold $(\widetilde{M}, \tilde{g})$, then (M, g) is isometric to $(\mathbb{S}^3_+, \bar{g})$.*

We recall that an isoperimetric surface is the boundary of an isoperimetric region, which is a domain $\Omega \subset M$ satisfying

$$|\partial\Omega| = \inf \{|\partial\Omega'| : \Omega' \subset M \text{ is a domain with } \mathrm{Vol}(\Omega') = \mathrm{Vol}(\Omega)\}.$$

Thus the hypotheses of the theorem require that any domain Ω in the doubled manifold $\widetilde{M}$ with $\mathrm{Vol}(\Omega) = \mathrm{Vol}(M)$ has area $|\partial\Omega| \geq |\partial M|$. This is a non-local boundary condition, in the sense that it depends on the behavior of (M, g) away from the boundary. Also of note is the fact that this theorem does not need the boundary to be isometric to $\mathbb{S}^2$, but instead only requires it to have the same area. (If $|\partial M| > 4\pi$, one could rescale g to obtain a new metric satisfying all the hypotheses of the theorem, with $|\partial M| = 4\pi$.)

The next positive result assumes a lower bound on Ricci curvature, rather than scalar curvature. This permits the use of Reilly's theorem on Dirichlet eigenvalues for manifolds with a lower curvature bound.

Theorem 16. *(Reilly (1977)) Let (M^n, g) be a compact Riemannian manifold with boundary Σ. Assume that $\mathrm{Ric}_g \geq (n-1)g$ and $H_g \geq 0$. Then $\lambda_1(-\Delta) \geq n$, with equality precisely when M is isometric to $\mathbb{S}^n_+$.*

Reilly's theorem, together with the Bochner formula, implies the following rigidity statement for manifolds with bounded Ricci curvature and convex boundary.

Theorem 17. *(Hang and Wang (2009)) Let (M^n, g) be a compact Riemannian manifold with boundary Σ. Suppose:*

- $\mathrm{Ric}_g \geq (n-1)g$;
- *(Σ, g) is isometric to $(\mathbb{S}^{n-1}, \bar{g})$;*

- Σ *is convex in* M.

Then M *is isometric to* $\mathbb{S}^n_+$.

We observe that this theorem requires convexity of the boundary, so the second fundamental form is assumed only to be nonnegative, rather than zero. This was also the case when $n = 2$, and when g was assumed to be conformal to $\bar{g}$.

Before stating the next result in our overview of scalar curvature rigidity theorems, we recall that an embedded, incompressible projective plane in a 3-manifold M is an embedded hypersurface $\Sigma \xrightarrow{i} M$ such that Σ is homeomorphic to $\mathbb{RP}^2$, and the induced map $i_* : \pi_1(\Sigma) \to \pi_1(M)$ is injective. We let

$$\mathcal{A}(M, g) := \inf\{|\Sigma| : \Sigma \text{ is an embedded, incompressible projective plane}\}.$$

Theorem 18. *(Bray, Brendle, Eichmair and Neves (2010)) Let* (M^3, g) *be a compact Riemannian manifold that contains an embedded, incompressible projective plane. Then*

$$\mathcal{A}(M, g) \inf R_g \le 12\pi,$$

with equality precisely when (M, g) *has constant sectional curvature.*

We complete our survey of positive results with a result on graphical hypersurfaces in Euclidean case. In the following theorem B denotes the open unit ball in $\mathbb{R}^n$.

Theorem 19. *(Huang and Wu (2010)) Suppose* $u \in C^2(B) \cap C^1(\bar{B})$, *and let* $M_u \subset \mathbb{R}^{n+1}$ *denote the graph of* u *over* $\bar{B}$. *If the induced metric on* M_u *has scalar curvature* $R \ge n(n-1)$, *then* M_u *is isometric to* $\mathbb{S}^n_+$.

We note that, in contrast to previous results, this theorem has no restrictions whatsoever on the boundary behavior of u. However, the requirement that M is a graphical hypersurface in $\mathbb{R}^n$ is rather stringent, so it is not surprising that less control of the boundary geometry is needed, It was recently discovered that Min-Oo's original conjecture is false, as demonstrated by the following theorem.

Theorem 20. *(Brendle, Marques and Neves (2010)) For any $n \geq 3$, there exists a smooth Riemannian metric g on the hemisphere $\mathbb{S}^n_+$ with the following properties:*

- $R_g \geq n(n-1)$ *in* $\mathbb{S}^n_+$*;*
- $R_g(p) > n(n-1)$ *for some point* $p \in \mathbb{S}^n_+$*;*
- g *agrees with* $\bar{g}$ *in a neighborhood of* $\partial\mathbb{S}^n_+$*.*

There are two main steps in the proof. First, a one-parameter family of metrics, $g(t)$, is constructed, with $R_{g(t)} > n(n-1)$ and $H_{g(t)} > 0$ for $t > 0$ sufficiently small. The family additionally satisfies $g(0) = \bar{g}$ and $g(t)(X,Y) = \bar{g}(X,Y)$ for all X and Y in $T\mathbb{S}^n|_{\partial\mathbb{S}^n_+}$. (This is stronger than the statement that g and $\bar{g}$ induce the same metric on the boundary, because it applies to the normal direction also.) Then a delicate gluing argument is applied to construct a metric that agrees with $\bar{g}$ near the boundary, and has the properties claimed by the above theorem.

In particular, we note that the first step of the construction outlined above implies that the Min–Oo conjecture—with the totally geodesic boundary condition replaced by the weaker constraint $H_g \geq 0$—is not even true for metrics that are C^2-close to the standard spherical metric. However, such a local result is true if we restrict our attention to a domain smaller than the entire hemisphere. For convenience we let f denote the restriction of the Euclidean coordinate function x^{n+1} to the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, and define $\Omega = \{f \geq c\}$, so that the case $c = 0$ corresponds to the entire upper hemisphere.

Theorem 21. *(Brendle and Marques (2010)) Let $\Omega = \{f \geq c\}$, where $c \geq \frac{2}{\sqrt{n+3}}$. Let g be a Riemannian metric on Ω with the following properties:*

- $R_g \geq n(n-1)$ *in* Ω*;*
- $H_g \geq H_{\bar{g}}$ *on* $\partial\Omega$*;*
- g *and* $\bar{g}$ *induce the same metric on* $\partial\Omega$*.*

If $g - \bar{g}$ *is sufficiently small in the* $W^{2,p}$ *norm, then* $g = \varphi^* \bar{g}$ *for some diffeomorphism* $\varphi : \Omega \to \Omega$ *with* $\varphi|_{\partial\Omega} = id$*.*

We thus see that local rigidity holds on sufficiently small geodesic balls, but fails on the entire hemisphere, and hence global rigidity also fails on the hemisphere.

The main goal of this dissertation is to investigate the following problems:

1. What is the largest ball on which the above local rigidity theorem is valid?
2. Can the theorem be extended to domains other than geodesic balls?
3. Is there any domain on which a global rigidity theorem is true?

Problems 1 and 2 are natural generalizations of Theorem 21. In particular, it would be interesting to find the exact point at which the geodesic balls $\{f \geq c\}$ lose their rigidity, and to understand the geometric significance of this critical domain. Since Theorem 21 demonstrated that geodesic balls are rigid only when of sufficiently small radius, it is reasonable to ask whether or not there is a global rigidity theorem on appropriately small domains. This is the motivation underlying Problem 3.

Progress toward these goals is summarized in the following section.

1.4 New results

The local rigidity theorem in Brendle and Marques (2010) is established by finding an upper bound for the integral

$$I := \int_\Omega f(R_g - \bar{R})d\bar{V},$$

where $\bar{R} = n(n-1)$ is the scalar curvature of the spherical metric, $\bar{g}$. The resulting bound involves terms proportional to the first two variations of the mean curvature H_g, evaluated at $\bar{g}$. Applying the hypothesis $H_g \geq \bar{H}$, it is then shown that I is negative when g is sufficiently close to $\bar{g}$. However, the curvature assumption $R_g \geq \bar{R}$ implies $I \geq 0$ and a contradiction is obtained, thus proving the local rigidity theorem.

Our first contribution is to recognize that the interior and boundary terms coming from I can combined into a single variational computation by considering the functional

$$\mathcal{F}_\Omega(g) := \int_\Omega f\left(R_g - \bar{R}\right) d\bar{V} + 2\int_\Sigma f|\bar{N}|_g^{-1}\left(H_g - \bar{H}\right) d\bar{\mu}.$$

This is clearly nonnegative when the hypotheses of the local rigidity theorem are satisfied, so it suffices to prove that, for appropriate domains, $\mathcal{F}_\Omega(g)$ is negative in some neighborhood of $\bar{g}$. To do so, we show that for certain choices of Ω, the spherical metric $\bar{g}$ is a nondegenerate critical point of $\mathcal{F}_\Omega$. It then follows from the Morse–Palais lemma that the behavior of $\mathcal{F}_\Omega$ near $\bar{g}$ is entirely determined by the Hessian, $D^2\mathcal{F}_\Omega(\bar{g})$.

Computing the second variation explicitly, we find that $D^2\mathcal{F}_\Omega(\bar{g})$ comprises interior terms, which are always negative, and boundary terms, which will be positive when Ω is large. We then show that the positive boundary terms can be controlled by the negative interior terms, via a weighted Sobolev trace-type inequality. This yields a improvement on the local rigidity statement of Brendle and Marques (2010), since it allows us to prove $D^2\mathcal{F}_\Omega(\bar{g})$ is negative definite even in the presence of positive boundary terms.

Theorem 22. *Suppose* $\Omega = \{f \geq c\} \subset \mathbb{S}^n$ *with*

$$c^2 \geq \begin{cases} \frac{2}{n+1} & \text{if } n \leq 5, \\ 4\left(\frac{4+n-\sqrt{2n-1}}{n^2+6n+17}\right) & \text{if } n > 5. \end{cases}$$

n	3	4	5	10	100	1000
$2/\sqrt{n+3}$	0.816	0.756	0.707	0.555	0.197	0.0632
c from Theorem 22	0.707	0.632	0.577	0.467	0.184	0.0618

Table 1.1: Comparison of Theorem 22 with the local rigidity theorem of Brendle and Marques (2010).

Then for $p > n$ there is a $W^{2,p}$ neighborhood U of $\bar{g}$ such that any $g \in U$ with $R_g \geq \bar{R}$, $H_g \geq \bar{H}$ and $g_\Sigma = \bar{g}_\Sigma$ is given by $g = \varphi^* \bar{g}$ for some diffeomorphism with $\varphi|_\Sigma = id$.

It is clear from the method of proof that this is always a strict improvement over the result of Brendle and Marques, which gives a lower bound $c \geq 2/\sqrt{n+3}$. For the sake of comparison we compute the respective lower bounds in a few dimensions; these are reported in Table 1.1. A simple computation shows that

$$\frac{4}{n+3} - 4\left(\frac{4+n-\sqrt{2n-1}}{n^2+6n+17}\right) \sim \frac{4\sqrt{2}}{n^{3/2}}$$

as $n \to \infty$. This gives a quantitative estimate of the amount by which the lower bound on c from Theorem 22 is an improvement over Brendle and Marques' lower bound, in the limit as n increases to infinity.

Our variational analysis of the functional $\mathcal{F}_\Omega(g)$ is sufficiently general as to allow for the study of domains other than geodesic balls. The geodesic balls $\{f \geq c\}$ considered above have totally umbilic boundary, so it is not surprising that some additional control on the geometry of the boundary is necessary in order to obtain a similar result for nonspherical domains. We thus assume that the mean curvature is sufficiently large compared to the smallest eigenvalue of the second fundamental form, as made precise by the following theorem.

Theorem 23. *Suppose $\Omega \subset \{f \geq c\} \subset \mathbb{S}^n$ has second fundamental form $\bar{A} \geq \lambda \bar{g}$ for* *some function* $\lambda \in C^0(\Sigma)$, *and mean curvature*

$$\bar{H} \geq \frac{1}{2c}\left(-\lambda + 5\sqrt{1-c^2} + \sqrt{\lambda^2 + 6\lambda\sqrt{1-c^2} + 17(1-c^2)}\right).$$

Then for any $p > n$ *there is a* $W^{2,p}$ *neighborhood* U *of* $\bar{g}$ *such that any* $g \in U$ *with* $R_g \geq \bar{R}$, $H_g \geq \bar{H}$ *and* $g_\Sigma = \bar{g}_\Sigma$ *is given by* $g = \varphi^*\bar{g}$ *for some diffeomorphism with* $\varphi|_\Sigma = id$.

It is important to note that the bound relating the mean curvature and second fundamental form is to be computed with respect to $\bar{g}$, so this is a requirement on the domain $\Omega \subset \mathbb{S}^n$, and has nothing to do with g. The condition on the boundary geometry of g is that $H_g \geq \bar{H}$, as before.

In the case that Σ is convex, hence $\bar{A} \geq 0$, the required lower bound on the mean curvature simplifies to

$$\bar{H} \geq \frac{5+\sqrt{17}}{2}\frac{\sqrt{1-c^2}}{c}.$$

It is also easy to see that this yields a positive result in the special case that Ω is a geodesic ball $\{f \geq c\}$, though the conclusion is somewhat weakened from that of Theorem 22 (which is of course less generally applicable). For $n = 3$ we find the lower bound $c \approx 0.843$.

Finally, we consider the eigenvalue problem associated to the quadratic form $B(h,h) := -D^2\mathcal{F}_\Omega(\bar{g})(h,h)$ on the geodesic ball $\Omega = \{f \geq c\}$. In particular, we show that for any $c \in (0,1)$ there exists $h \in W^{1,2}(\Omega)$ satisfying

$$B(h,h) = \mu(c) := \inf\{B(h',h') : h' \in W^{1,2}\},$$

and that h is in fact a smooth solution of an elliptic boundary-value problem. We prove that $\mu : [0,1) \to \mathbb{R}$ is continuous, then conclude our discussion of local rigidity with the following Morse-theoretic conjecture.

Conjecture 24. *There exist constants $c_1 > c_2 > \cdots > c_N$ such that the domain $\{f \geq c\}$ is nondegenerate for all $c \in [0, 1) \setminus \{c_1, \ldots, c_N\}$. Moreover, the index of the quadratic form B is given by*

$$\textit{ind } B(c) = \max\{i : c_i > c\}.$$

This says that the index of B is given by the number of degenerate geodesic balls strictly contained in $\{f \geq c\}$, and that B is positive definite if and only if $c > c_1$.

We next turn our attention to potential global rigidity statements, in particular focusing on the question of whether or not Min–Oo's conjecture is true on some domain smaller than the hemisphere. We perform a blowup analysis by assuming the existence of counterexamples on arbitrarily small domains, which are then rescaled to produce a sequence of metrics on the unit ball in $\mathbb{R}^n$. If the original sequence of counterexamples satisfies some uniform geometric bounds, it is possible to extract a convergent subsequence from the rescaled sequence of metrics—all of which have nonnegative scalar curvature and boundary behavior approaching that of the standard Euclidean metric.

The idea is then to prove the limiting metric is flat, hence the original sequence is converging to the standard spherical metric in an appropriately strong topology. This would eventually produce a counterexample to the local version of Min–Oo's conjecture, thus yielding a contradiction.

That the limit is flat seems like a straightforward application of the positive mass theorem. However, the subsequence that we find is only converging in $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, so the limiting metric may not even have well-defined scalar curvature. To address this issue, we prove in Chapter 4 a version of the positive mass theorem for compact manifolds with boundary (similar to the results of Miao and Shi–Tam), for metrics that are *a priori* only of class C^1.

In Chapter 5 we use our C^1 rigidity theorem to prove that the limiting metric of the above-constructed subsequence is indeed flat, and that the convergence occurs in $W^{2,p}$ for any $p \geq 1$. Rescaling back to the original sequence of counterexamples on $\mathbb{S}^n$, we find that these converge to the standard spherical metric in $W^{2,p}$ for any $1 \leq p \leq \frac{n}{2}$. While this convergence is not strong enough to permit application of the local rigidity theorem, it is interesting nonetheless, and suggests future directions to explore in pursuit of global rigidity.

1.5 Definitions and notation

We adopt the convention that a tensor field *of type* (p, q) has p contravariant and q covariant indices. Thus vector fields and one-forms are of type $(1, 0)$ and $(0, 1)$, respectively. Given a Riemannian metric g with the corresponding Levi–Civita connection ∇, we define the *curvature operator* by the formula

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any vector fields X, Y and Z. The *Riemann curvature tensor* is the $(0, 4)$-tensor given by

$$\mathrm{Rm}(Z, Y, Z, W) := \langle R(X, Y)Z, W\rangle.$$

Following this sign convention, the *Ricci tensor* Ric is the symmetric $(0, 2)$-tensor obtained by tracing Rm in the first and last indices. We write this as $\mathrm{Ric} = \mathrm{tr}_{14}\,\mathrm{Rm}$. In local coordinates $(x^1, \ldots, x^n)$ we have $R_{jk} := \mathrm{Ric}(\partial_j, \partial_k) = g^{il} R_{ijkl}$. where $R_{ijkl} := \mathrm{Rm}(\partial_i, \partial_j, \partial_k, \partial_l)$. The components of the curvature tensor can be computed explicitly in terms of the *Christoffel symbols*

$$\Gamma_{ij}^k := \frac{g^{kl}}{2} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right). \tag{1.2}$$

We will make frequent use of the following lemma.

Lemma 25. *Each tensor field h of type $(0, 2)$ satisfies the commutation relation*

$$\nabla_i \nabla_j h_{kl} = \nabla_j \nabla_i h_{kl} - R_{ijk}{}^m h_{ml} - R_{ijl}{}^m h_{km}.$$

The *divergence* of a (p,q)-tensor A is the $(p,q-1)$-tensor $\delta A := -\operatorname{tr}_{12} \nabla A$. The formal L^2-adjoint of δ is the covariant derivative ∇. The restriction of δ to the space of symmetric $(0,2)$-tensors has formal adjoint

$$\delta^* \omega = \frac{1}{2} \mathcal{L}_{\omega^\sharp} g, \tag{1.3}$$

where $\omega^\#$ denotes the vector field dual to the one-form ω. We define the *matrix product* of $(0,2)$-tensors by $A \times B := \operatorname{tr}_{23} A \otimes B$.

Lemma 26. *Let h by a symmetric $(0,2)$-tensor field. Then*

$$\delta^2(h \times h) = \nabla^k h^{ij} \nabla_i h_{jk} - 2\langle h, \nabla(\delta h)\rangle + |\delta h|^2 + \langle h \times h, \mathrm{Ric}\rangle - R_{ijkl} h^{il} h^{jk}.$$

Proof. In local coordinates we have $(h \times h)_{jl} = g^{pq} h_{jp} h_{ql}$. We thus compute

$$\begin{aligned}
\delta^2(h \times h) &= g^{il} g^{jk} g^{pq} \nabla_i \nabla_k h_{jp} h_{ql} \\
&= g^{il} g^{jk} g^{pq} (h_{ql} \nabla_i \nabla_k h_{jp} + \nabla_i h_{ql} \nabla_k h_{jp} + \nabla_i h_{jp} \nabla_k h_{ql} + h_{jp} \nabla_i \nabla_k h_{ql}).
\end{aligned}$$

We immediately recognize the first two terms as $-\langle h, \nabla(\delta h)\rangle$ and $|\delta h|^2$, respectively. To compute the fourth term we use Lemma 1.3 to obtain

$$\begin{aligned}
g^{il} g^{jk} g^{pq} h_{jp} \nabla_i \nabla_k h_{ql} &= g^{il} h^{kq} (\nabla_k \nabla_i h_{ql} - R_{ikq}{}^m h_{ml} - R_{ikl}{}^m h_{qm}) \\
&= -\langle h, \nabla(\delta h)\rangle - R_{ikqm} h^{mi} h^{kq} + \langle h \times h, \mathrm{Ric}\rangle.
\end{aligned}$$

The proof follows after reindexing. □

Abusing notation, we denote the divergence of a vector field, X, by δX. (In the above notation this would be written $\delta(X^\flat)$.) Now suppose M is a compact, oriented Riemannian manifold with boundary ∂M, and *inward* unit normal N. With our sign conventions, the divergence theorem says

$$\int_M \delta X = \int_{\partial M} \langle X, N\rangle \tag{1.4}$$

for any vector field X on M.

We next recall some variational formulae for evolving Riemannian metrics.

Lemma 27. *Let $g(t)$ be a smooth one-parameter family of Riemannian metrics moving with speed h. The following formulae hold:*

$$\frac{\partial}{\partial t} g^{ij} = -h^{ij}; \tag{1.5}$$

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{g^{kl}}{2}\left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\right); \tag{1.6}$$

$$\begin{aligned} 2\frac{\partial}{\partial t} R_{ijk}{}^l = {} & g^{lm}\left(\nabla_i\nabla_k h_{jm} + \nabla_j\nabla_m h_{ik} - \nabla_i\nabla_m h_{jk} - \nabla_j\nabla_k h_{im}\right) \\ & - g^{lp}\left(R_{ijk}{}^q h_{pq} + R_{ijp}{}^q h_{kq}\right); \end{aligned} \tag{1.7}$$

$$-2\frac{\partial}{\partial t} R_{jk} = \Delta_L h_{jk} + \nabla_j\nabla_k(\operatorname{tr} h) + \nabla_j(\delta h)_k + \nabla_k(\delta h)_j; \tag{1.8}$$

$$\frac{\partial}{\partial t} R = \delta^2 h - \Delta(\operatorname{tr} h) - \langle h, \mathrm{Ric}\rangle. \tag{1.9}$$

In the penultimate formula, Δ_L denotes the *Lichnerowicz Laplacian*, defined by

$$\Delta_L h_{jk} := \Delta h_{jk} + 2R_{ijkl}h^{il} - g^{il}\left(R_{ij}h_{kl} + R_{kl}h_{ij}\right)$$

for any symmetric $(0,2)$-tensor field h.

We finally discuss the geometry of submanifolds. Suppose $\Sigma \subset M$ is an orientable hypersurface, with unit normal N. We define the *second fundamental form* A by

$$A(X,Y) := \langle \nabla_X Y, N\rangle$$

for all tangential vector fields X and Y. This implies $\nabla_X Y = \nabla_X^\Sigma Y + A(X,Y)N$, where ∇^Σ denotes the Levi–Civita connection of the induced metric on Σ. We then define the *mean curvature* $H := \operatorname{tr}_\Sigma A$. When Σ is the boundary of a domain $\Omega \subset M$, we choose N to be the inward unit normal. With these conventions the unit sphere in $\mathbb{R}^n$ has mean curvature $n-1$.

For any symmetric $(0,2)$-tensor T and vector field X, we let $T\cdot X := T(X,\cdot)^\sharp$ denote the vector field dual to the one-form $T(X,\cdot)$. We then have the following simple but useful observation.

Lemma 28. *For every tangential vector field X we have*

$$\nabla_X N = -A \cdot X.$$

The proof follows immediately from the definition of A. We finish with a coordinate construction that will prove useful in computing the first and second variation of the second fundamental form.

Lemma 29. *Let $p \in M$. There exists an orthonormal frame $(E_1, \ldots, E_n)$ in a neighborhood U of p in M such that*

$$\begin{aligned} E_\alpha\big|_p &\text{ is tangent to } \Sigma \text{ for } 1 \leq \alpha \leq n-1, \\ E_n\big|_p &\text{ is normal to } \Sigma, \\ (\nabla E_i)(p) = 0 &\text{ for all } i. \end{aligned}$$

Such a frame will be called an *adapted geodesic normal frame at p*. To construct such a frame, we simply choose orthonormal frame fields $\{\tilde{E}_i\}$ with $\left(\nabla \tilde{E}_i\right)(p) = 0$, and apply a constant rotation, $E_j = \Lambda^i_j \tilde{E}_i$, so that $E_n = N$ at p.

Finally, letting j denote the inclusion of Σ into M, we define for any tensor T on M the tangential restriction $T_\Sigma := j^* T$. Note that this only contains the tangential part of T, so it is a section of the tensor bundle of $T\Sigma \to \Sigma$, rather than $TM \to \Sigma$.

2

A functional inequality for the scalar and mean curvature

In this chapter we define a functional $\mathcal{F}$ of Riemannian metrics on a domain $\Omega \subset \mathbb{S}^n$, incorporating both the scalar curvature in the interior and the mean curvature on the boundary. Through a series of lengthy variational computations, we find the first and second variation of $\mathcal{F}$. It is observed that the standard spherical metric, $\bar{g}$ is a critical point for $\mathcal{F}$. We then apply the Morse–Palais lemma, together with a suitable version of Ebin's slice theorem, to provide an exact formula for $\mathcal{F}(g)$ for metrics g that are suitably close to $\bar{g}$. This expression, given by Theorem 30, will be used in Chapter 3 to prove rigidity theorems for certain classes of domains.

2.1 The weighted scalar curvature functional

Let $f := x^{n+1}|_{\mathbb{S}^n}$ denote the restriction to the sphere of the Euclidean coordinate function x^{n+1}. For a domain $\Omega \subset \mathbb{S}^n$ with smooth boundary Σ, we define the *weighted scalar curvature functional* by

$$\mathcal{F}_\Omega(g) := \int_\Omega f \left(R_g - \bar{R}\right) d\bar{V} + 2 \int_\Sigma f |\bar{N}|_g^{-1} \left(H_g - \bar{H}\right) d\bar{\mu} \tag{2.1}$$

for any Riemannian metric g on Ω. Here $d\bar{V}$ denotes the volume form of the standard spherical metric, $\bar{g}$, and $d\bar{\mu}$ the corresponding area form on Σ. Additionally, $\bar{N}$ is the inward unit normal computed with respect to $\bar{g}$. The $\bar{R}$ term in the first integral is constant (independent of g); it is included so that $\mathcal{F}_\Omega(\bar{g}) = 0$.

We say that a domain Ω is *nondegenerate* if the bilinear form $D^2\mathcal{F}_\Omega(\bar{g})(h, h)$ is nondegenerate on the space $\{h \in W^{2,p} : \bar{\delta} h = 0 \text{ and } h_\Sigma = 0\}$ for some $p > n$.

Theorem 30. *Suppose $\Omega \subset \mathbb{S}^n$ is a nondegenerate domain. If g and $\bar{g}$ induce the same metric on Σ and $\|g - \bar{g}\|_{W^{2,p}(\Omega,\bar{g})}$ is sufficiently small, then there exists a diffeomorphism $\varphi : \Omega \to \Omega$ that fixes Σ, such that*

$$\begin{aligned} 2\mathcal{F}_\Omega(\varphi^* g) = &- \int_\Omega f \left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}(d \operatorname{tr} h)^2 + |h|^2 + (\operatorname{tr} h)^2 \right) \\ &+ \int_\Sigma \left[-2h(N, N)h(\nabla^\Sigma f, N) + \big(h(N, N)^2 + (h \times h)(N, N) \big) \nabla_N f \right] \\ &+ \int_\Sigma f \left[\frac{1}{2} h(N, N)^2 H - (h \times h)(N, N) H - \langle (h \times h)_\Sigma, A \rangle \right], \end{aligned}$$

where h is a symmetric tensor satisfying $h_\Sigma = 0$, $\bar{\delta} h = 0$ and $\|h\|_{W^{2,p}(\Omega,\bar{g})} \le C\|g - \bar{g}\|_{W^{2,p}(\Omega,\bar{g})}$ for some constant $C = C(\Omega, p)$.

In the following chapter we will investigate the nondegeneracy of domains, and its relation to scalar curvature rigidity. We also note the following special case of Theorem 30 for geodesic balls centered at the north pole.

Corollary 31. *Suppose $\Omega = \{f \ge c\}$, with all other hypotheses as above. Then*

$$\begin{aligned} 2\mathcal{F}_\Omega(g) = &- \int_\Omega f \left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}(d \operatorname{tr} h)^2 + |h|^2 + (\operatorname{tr} h)^2 \right) \\ &+ \left(\sqrt{1 - c^2} - \frac{nc^2}{\sqrt{1 - c^2}} \right) \int_\Sigma (h \times h)(N, N) \\ &+ \left(\sqrt{1 - c^2} + \frac{(n+1)c^2}{2\sqrt{1 - c^2}} \right) \int_\Sigma h(N, N)^2. \end{aligned}$$

The remainder of the chapter is devoted to the proof of Theorem 30; we begin by explicitly computing the first and second variations of the scalar curvature and mean curvature terms arising in the definition of $\mathcal{F}_\Omega$.

2.2 Scalar curvature variations

We denote the first and second derivatives of the scalar curvature operator, $g \mapsto R(g)$ by L_g and Q_g, respectively. (The chosen letters correspond to "linear" and "quadratic".) Explicitly, these are defined as

$$\mathrm{L}_g(h) := \left.\frac{d}{dt}\right|_{t=0} R(g+th),$$

$$\mathrm{Q}_g(h) := \left.\frac{d^2}{dt^2}\right|_{t=0} R(g+th).$$

It was observed in Section 1.5 that $\mathrm{L}_g(h) = \delta^2 h - \Delta(\operatorname{tr} h) - \langle h, \mathrm{Ric}\rangle$. In this section we give an explicit formula for the second variation, $\mathrm{Q}_g(h)$. This formula was stated without proof in Fischer and Marsden (1975).

Proposition 32. *The second variation of scalar curvature is given by*

$$\begin{aligned} \mathrm{Q}_g(h) = &-\frac{1}{2}|\nabla h|^2 + 2\langle h \times h, \mathrm{Ric}\rangle - \frac{1}{2}(d\operatorname{tr} h)^2 + \nabla^k h^{ij}\nabla_i h_{jk} \\ &+ 2\langle h, \mathrm{Hess}(\operatorname{tr} h)\rangle - 2\langle \delta h, d(\operatorname{tr} h)\rangle + \Delta|h|^2 - 2\delta^2(h\times h). \end{aligned}$$

The proof consists of a series of elementary, though tedious, variational computations. We assume throughout that $g(t)$ is a smooth one-parameter family of Riemannian metrics evolving with speed h. For convenience we define $G(h) := h - \frac{1}{2}(\operatorname{tr} h)g$. It follows immediately that $\delta G(h) = \delta h + \frac{1}{2}d(\operatorname{tr} h)$.

Lemma 33. *Let $f(t)$ be a smooth one-parameter family of functions. Then*

$$\frac{\partial}{\partial t}\Delta f = \Delta\left(\frac{\partial f}{\partial t}\right) - \langle h, \mathrm{Hess}\, f\rangle + \langle \delta G(h), df\rangle.$$

Proof. In local coordinates we have $\Delta f = g^{ij}\nabla_i\nabla_j f = g^{ij}(\partial_i\partial_j f - \Gamma^k_{ij}\partial_k f)$. Differentiating, we obtain

$$\frac{\partial}{\partial t}\Delta f = -h^{ij}\left(\partial_i\partial_j f - \Gamma^k_{ij}\partial_k f\right) + g^{ij}\left(\partial_i\partial_j\left(\frac{\partial f}{\partial t}\right) - \Gamma^k_{ij}\partial_k\left(\frac{\partial f}{\partial t}\right) - \left(\frac{\partial \Gamma^k_{ij}}{\partial t}\right)\partial_k f\right)$$

$$= \Delta\left(\frac{\partial f}{\partial t}\right) - \langle h, \operatorname{Hess} f\rangle - g^{ij}\frac{\partial \Gamma^k_{ij}}{\partial t}\partial_k f.$$

By Lemma 27 the final term is equal to

$$-\frac{g^{ij}g^{kl}}{2}\left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\right)\partial_k f = \langle \delta h, df\rangle + \frac{1}{2}\langle d(\operatorname{tr} h), df\rangle.$$

This completes the proof. □

We next compute the evolution of the divergence operator acting on one-forms.

Lemma 34. *Let $\omega(t)$ be a smooth one-parameter family of one-forms. Then*

$$\frac{\partial}{\partial t}\delta\omega = \delta\left(\frac{\partial \omega}{\partial t}\right) + \langle h, \nabla\omega\rangle - \langle \delta G(h), \omega\rangle.$$

Proof. We begin with the local formula $\delta\omega = -g^{ij}\nabla_i\omega_j = -g^{ij}(\partial_i\omega_j - \Gamma^k_{ij}\omega_k)$ and differentiate as in the proof of Lemma 33 to obtain

$$\frac{\partial}{\partial t}\delta\omega = \delta\left(\frac{\partial \omega}{\partial t}\right) + h^{ij}\nabla_i\omega_j - g^{ij}\frac{\partial \Gamma^k_{ij}}{\partial t}\omega_k.$$

We complete the proof by applying Lemma 27 to see that

$$g^{ij}\frac{\partial \Gamma^k_{ij}}{\partial t}\omega_k = \langle \delta G(h), \omega\rangle.$$

□

Finally, we examine the divergence and trace of $(0,2)$-tensors.

Lemma 35. *Let $A(t)$ be a smooth one-parameter family of symmetric $(0,2)$-tensors. Then*

$$\frac{\partial}{\partial t}\operatorname{tr} A = \operatorname{tr}\left(\frac{\partial A}{\partial t}\right) - \langle h, A\rangle$$

and

$$\frac{\partial}{\partial t}\delta A = \delta\left(\frac{\partial A}{\partial t}\right) - \delta(h \times A) - \frac{1}{2}A\left(d(\operatorname{tr} h), \cdot\right) + \frac{A^{ij}}{2}\nabla_k h_{ij}.$$

Proof. The first statement follows immediately upon differentiating $\operatorname{tr} A = g^{ij}A_{ij}$. To prove the second statement, we begin with the local coordinate expression $(\delta A)_k = -g^{ij}\nabla_i A_{jk} = -g^{ij}(\partial_i A_{jk} - \Gamma^l_{ij}A_{kl} - \Gamma^l_{ik}A_{jl})$. As in the proof of Lemma 33, we differentiate this to obtain

$$\frac{\partial}{\partial t}(\delta A)_k = \delta\left(\frac{\partial A}{\partial t}\right) + h^{ij}\nabla_i A_{jk} + g^{ij}\frac{\partial \Gamma^l_{ij}}{\partial t}A_{kl} + g^{ij}\frac{\partial \Gamma^l_{ik}}{\partial t}A_{jl},$$

and observe that

$$g^{ij}\frac{\partial \Gamma^l_{ij}}{\partial t}A_{kl} = -A\left(\delta G(h), \cdot\right).$$

By the Leibniz rule we have

$$\begin{aligned} h^{ij}\nabla_i A_{jk} &= \nabla_i h^{ij} A_{jk} - A_{jk}\nabla_i h^{ij} \\ &= -\delta(h \times A) + A(\delta h, \cdot). \end{aligned}$$

The final term is

$$\frac{g^{ij}g^{lm}}{2}\left(\nabla_i h_{km} + \nabla_k h_{im} - \nabla_m h_{ik}\right)A_{jl} = \frac{A^{ij}}{2}\nabla_k h_{ij},$$

where we have used the symmetry of h and A to cancel the first and last terms in parentheses. The statement now follows from combining these expressions. □

With these lemmata at our disposal we can differentiate the first term in $\mathrm{L}_g(h)$.

Lemma 36. *Let $g(t) = g + th$. Then*

$$\frac{\partial}{\partial t}\delta^2 h = -\frac{1}{2}|\nabla h|^2 + \frac{1}{2}\langle h, \mathrm{Hess}(\mathrm{tr}\, h)\rangle - \langle \delta h, d(\mathrm{tr}\, h)\rangle$$
$$- \delta^2(h \times h) - \frac{1}{2}\langle h, \Delta h\rangle + \langle h, \nabla(\delta h)\rangle - |\delta h|^2$$

Proof. We first apply Lemma 34 to the one-form $\omega = \delta h$, to obtain

$$\frac{\partial}{\partial t}\delta^2 h = \delta\left(\frac{\partial}{\partial t}\delta h\right) + \langle h, \nabla(\delta h)\rangle - \langle \delta G(h), \delta h\rangle.$$

We then use Lemma 35, with $A = h$, to compute

$$\frac{\partial}{\partial t}\delta h = -\delta(h \times h) - \frac{1}{2}h\left(d(\mathrm{tr}\, h), \cdot\right) + \frac{h^{ij}}{2}\nabla_k h_{ij}.$$

In local coordinates we have $h\left(d(\mathrm{tr}\, h), \cdot\right)_l = g^{ij}h_{kl}\nabla_k h_{ij}$. The divergence of this term is then given by

$$-g^{ij}(\nabla^l h_{kl}\nabla_k h_{ij} + h_{kl}\nabla^l\nabla_k h_{ij}) = \langle \delta h, d(\mathrm{tr}\, h)\rangle - \langle h, \mathrm{Hess}(\mathrm{tr}\, h)\rangle.$$

The divergence of $h^{ij}\nabla_k h_{ij}$ is

$$-\nabla^k h^{ij}\nabla_k h_{ij} - h^{ij}\nabla^k\nabla_k h_{ij} = -|\nabla h|^2 - \langle h, \Delta h\rangle,$$

so we see that

$$\delta\left(\frac{\partial}{\partial t}\delta h\right) = -\delta^2(h \times h) - \frac{1}{2}\langle \delta h, d(\mathrm{tr}\, h)\rangle + \frac{1}{2}\langle h, \mathrm{Hess}(\mathrm{tr}\, h)\rangle - \frac{1}{2}|\nabla h|^2 - \frac{1}{2}\langle h, \Delta h\rangle.$$

The proof follows. □

We next deal with the $\Delta(\mathrm{tr}\, h)$ term occurring in $\mathrm{L}_g(h)$.

Lemma 37. *Let $g(t) = g + th$. Then*

$$\frac{\partial}{\partial t}\Delta(\mathrm{tr}\, h) = -\Delta|h|^2 - \langle h, \mathrm{Hess}(\mathrm{tr}\, h)\rangle + \langle \delta G(h), d(\mathrm{tr}\, h)\rangle.$$

Proof. We let $f = \operatorname{tr} h$ and apply Lemma 33 to obtain

$$\frac{\partial}{\partial t}\Delta(\operatorname{tr} h) = \Delta\left(\frac{\partial}{\partial t}\operatorname{tr} h\right) - \langle h, \operatorname{Hess}(\operatorname{tr} h)\rangle + \langle \delta G(h), d(\operatorname{tr} h)\rangle.$$

Now Lemma 35, with $A = h$, implies

$$\frac{\partial}{\partial t}\operatorname{tr} h = -|h|^2.$$

The result follows. □

The last ingredient in the proof of Proposition 32 is the variation of the curvature term, $\langle h, \operatorname{Ric}\rangle$.

Lemma 38. *Let $g(t) = g + th$. Then*

$$\begin{aligned}\frac{\partial}{\partial t}\langle h, \operatorname{Ric}\rangle = &-\frac{1}{2}\langle h, \Delta h\rangle - \frac{1}{2}\langle h, \operatorname{Hess}(\operatorname{tr} h)\rangle - \langle h, \nabla(\delta h)\rangle \\ &- \langle h \times h, \operatorname{Ric}\rangle - R_{ijkl}h^{il}h^{jk}.\end{aligned}$$

Proof. Using local coordinates we have $\langle h, \operatorname{Ric}\rangle = h^{ij}R_{ij}$. It follows from the equation $h^{ij} = g^{ik}g^{jl}h_{kl}$ that

$$\begin{aligned}\frac{\partial}{\partial t}h^{ij} &= -h^{ik}g^{jl}h_{kl} - g^{ik}h^{jl}h_{kl} \\ &= -2(h \times h)^{ij},\end{aligned}$$

hence

$$\frac{\partial}{\partial t}\langle h, \operatorname{Ric}\rangle = -2\langle h \times h, \operatorname{Ric}\rangle + \langle h, \frac{\partial}{\partial t}\operatorname{Ric}\rangle.$$

It follows from Lemma 27 that

$$-2\langle h, \frac{\partial}{\partial t}\operatorname{Ric}\rangle = \langle h, \Delta_L h\rangle + \langle h, \operatorname{Hess}(\operatorname{tr} h)\rangle + 2\langle h, \nabla(\delta h)\rangle.$$

We also see that

$$\langle h, \Delta_L h\rangle = \langle h, \Delta h\rangle + 2R_{ijkl}h^{il}h^{jk} - g^{il}h^{jk}\left(R_{ij}h_{kl} + R_{kl}h_{ij}\right)$$
$$= \langle h, \Delta h\rangle + 2R_{ijkl}h^{il}h^{jk} - 2\langle h \times h, \mathrm{Ric}\rangle$$

and the result follows. □

We are now prepared to complete the proof of Proposition 32.

Proof. We recall the formula for the first variation, $\mathrm{L}_g(h) = \delta^2 h - \Delta(\mathrm{tr}\, h) - \langle h, \mathrm{Ric}\rangle$, and combine the results of Lemma 36, Lemma 37 and Lemma 38 to obtain

$$\begin{aligned} \mathrm{Q}_g(h) = &-\frac{1}{2}|\nabla h|^2 + \langle h \times h, \mathrm{Ric}\rangle - \frac{1}{2}(d\,\mathrm{tr}\, h)^2 + 2\langle h, \mathrm{Hess}(\mathrm{tr}\, h)\rangle \\ &- 2\langle \delta h, d(\mathrm{tr}\, h)\rangle + \Delta|h|^2 - \delta^2(h \times h) \\ &+ 2\langle h, \nabla(\delta h)\rangle - |\delta h|^2 + R_{ijkl}h^{il}h^{jk}. \end{aligned} \tag{2.2}$$

Using Lemma 26, we rewrite the last three terms:

$$2\langle h, \nabla(\delta h)\rangle - |\delta h|^2 + R_{ijkl}h^{il}h^{jk} = \nabla^k h^{ij}\nabla_i h_{jk} - \delta^2(h \times h) + \langle h \times h, \mathrm{Ric}\rangle.$$

This completes the proof. □

2.3 Mean curvature variations

In this section we derive formulae for the first and second variation of the second fundamental form and mean curvature of a fixed hypersurface $\Sigma \subset M$, assuming that the induced metric on Σ is constant. The main result of this section is the following.

Proposition 39. *Let $g(t) = g + th$, with $h_\Sigma = 0$ and $\delta_g h = 0$. Then*

$$\frac{\partial H}{\partial t} = -\frac{1}{2}\delta_\Sigma(h \cdot N) - \frac{1}{2}\nabla_N(\mathrm{tr}\, h)$$

and

$$\frac{\partial^2 H}{\partial t^2} = \left[(h \times h)(N, N) - \frac{3}{4}h(N, N)^2\right] H + \frac{1}{2}h(N, N)\left[\delta_\Sigma(h \cdot N) + \nabla_N(\mathrm{tr}\, h)\right].$$

at $t = 0$.

In this section we frequently use adapted geodesic normal frames, as given by Lemma 29. For the evolving family of metrics $g(t) = g + th$, Lemma 27 implies

$$\left\langle \frac{\partial}{\partial t} \nabla_X Y, Z \right\rangle = \frac{1}{2} \left[(\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z) - (\nabla_Z h)(X, Y) \right]$$

for any time-independent vector fields X, Y and Z. When X and Y are time-dependent, we instead write

$$\left\langle \frac{\partial}{\partial t} \nabla, Z \right\rangle (X, Y) = \frac{1}{2} \left[(\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z) - (\nabla_Z h)(X, Y) \right]$$

to avoid confusion.

Lemma 40. *If X is tangential to Σ then*

$$\left\langle \frac{\partial}{\partial t} \nabla, X \right\rangle = \left\langle \frac{\partial}{\partial t} \nabla^{\Sigma}, X \right\rangle - h(N, X) A.$$

Proof. Computing with respect to an adapted geodesic normal frame at (p_0, t_0), we have

$$\begin{aligned} 2 \left\langle \frac{\partial}{\partial t} \nabla, E_\gamma \right\rangle (E_\alpha, E_\beta) &= (\nabla_\alpha h)(E_\beta, E_\gamma) + (\nabla_\beta h)(E_\alpha, E_\gamma) - (\nabla_\gamma h)(E_\alpha, E_\beta) \\ &= \nabla_\alpha (h(E_\beta, E_\gamma)) - h(\nabla_\alpha E_\beta, E_\gamma) - h(E_\beta, \nabla_\alpha E_\gamma) \\ &+ \nabla_\beta (h(E_\alpha, E_\gamma)) - h(\nabla_\beta E_\alpha, E_\gamma) - h(E_\alpha, \nabla_\beta E_\gamma) \\ &- \nabla_\gamma (h(E_\alpha, E_\beta)) + h(\nabla_\gamma E_\alpha, E_\beta) + h(E_\alpha, \nabla_\gamma E_\beta) \\ &= (\nabla^{\Sigma}_\alpha h)(E_\beta, E_\gamma) + (\nabla^{\Sigma}_\beta h)(E_\alpha, E_\gamma) - (\nabla^{\Sigma}_\gamma h)(E_\alpha, E_\beta) \\ &- 2h(N, E_\gamma) A(E_\alpha, E_\beta) \end{aligned}$$

as claimed. □

Lemma 41. *For time-independent vector fields X, Y and Z we have*

$$\frac{\partial}{\partial t} \left\langle \frac{\partial}{\partial t} \nabla_X Y, Z \right\rangle = - \left\langle \frac{\partial}{\partial t} \nabla_X Y, h \cdot Z \right\rangle.$$

Proof. We start by writing

$$2\frac{\partial}{\partial t}\left\langle \frac{\partial}{\partial t}\nabla_X Y, Z\right\rangle = Xh(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$
$$+ Yh(X,Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z)$$
$$- Zh(X,Y) + h(\nabla_Z X, Y) + h(X, \nabla_Z Y).$$

Since $Xh(Y,Z)$ and $\nabla_X Y - \nabla_Y X = [X,Y]$ are time-independent (and similarly for all other permutations), we see that

$$\frac{\partial}{\partial t}\left\langle \frac{\partial}{\partial t}\nabla_X Y, Z\right\rangle = -\frac{\partial}{\partial t}h(\nabla_X Y, Z)$$
$$= -h\left(\frac{\partial}{\partial t}\nabla_X Y, Z\right)$$
$$= -\left\langle \frac{\partial}{\partial t}\nabla_X Y, h\cdot Z\right\rangle$$

as claimed. □

When Z is time-dependent, this becomes

$$\frac{\partial}{\partial t}\left\langle \frac{\partial}{\partial t}\nabla_X Y, Z\right\rangle = \left\langle \frac{\partial}{\partial t}\nabla_X Y, \frac{\partial Z}{\partial t} - h\cdot Z\right\rangle.$$

We thus obtain the following variational formulae for the second fundamental form.

Proposition 42. *Let $g(t) = g + th$. Then*

$$\frac{\partial A}{\partial t} = \frac{1}{2}h(N,N)A + \left\langle \frac{\partial}{\partial t}\nabla, N\right\rangle$$

and

$$\frac{\partial^2 A}{\partial t^2} = \left[(h\times h)(N,N) - \frac{5}{4}h(N,N)^2\right]A$$
$$- h(N,N)\left\langle \frac{\partial}{\partial t}\nabla, N\right\rangle - 2\left\langle \frac{\partial}{\partial t}\nabla^\Sigma, (h\cdot N)_\Sigma\right\rangle.$$

We note that this result requires no boundary assumptions on h.

Proof. For any tangential vector fields X and Y we have

$$\frac{\partial A}{\partial t}(X,Y) = h(\nabla_X Y, N) + \left\langle \frac{\partial}{\partial t}\nabla_X Y, N \right\rangle + \left\langle \nabla_X Y, \frac{\partial}{\partial t} N \right\rangle.$$

Evaluating in an adapted geodesic normal frame at (p_0, t_0), we find the tensorial expression

$$\frac{\partial A}{\partial t}(E_\alpha, E_\beta) = \left[h(N,N) + \left\langle N, \frac{\partial}{\partial t} N \right\rangle \right] A(E_\alpha, E_\beta) + \left\langle \frac{\partial}{\partial t}\nabla_\alpha E_\beta, N \right\rangle.$$

We then differentiate the equation $|N|^2 = 1$ to obtain

$$2\left\langle N, \frac{\partial}{\partial t} N \right\rangle + h(N,N) = 0,$$

and the first result follows.

For the second equation, we differentiate the above result and apply Lemma 41 to obtain

$$\frac{\partial^2 A}{\partial t^2} = \left[h\left(N, \frac{\partial}{\partial t} N\right) + \frac{1}{4} h(N,N)^2 \right] A + \left\langle \frac{\partial}{\partial t}\nabla, \frac{\partial N}{\partial t} - h \cdot N + \frac{1}{2} h(N,N) N \right\rangle.$$

Again using an adapted geodesic normal frame at (p_0, t_0), we have $\langle N, E_\alpha \rangle = 0$ for all t, hence

$$\left\langle \frac{\partial N}{\partial t}, E_\alpha \right\rangle = -h(N, E_\alpha).$$

We then can write at time t_0

$$\begin{aligned} \frac{\partial N}{\partial t} &= \left\langle N, \frac{\partial}{\partial t} N \right\rangle N + \sum_\alpha \left\langle \frac{\partial N}{\partial t}, E_\alpha \right\rangle E_\alpha \\ &= -\frac{1}{2} h(N,N) N - \sum_\alpha h(N, E_\alpha) E_\alpha \end{aligned}$$

and similarly

$$h \cdot N = h(N,N)N + \sum_\alpha h(N, E_\alpha)E_\alpha.$$

These formulae imply

$$\frac{\partial N}{\partial t} - h \cdot N + \frac{1}{2}h(N,N)N = -h(N,N)N - 2\sum_\alpha h(N,E_\alpha)E_\alpha$$

and

$$\begin{aligned} h\left(N, \frac{\partial}{\partial t}N\right) &= -\frac{1}{2}h(N,N)^2 - \sum_\alpha h(N,E_\alpha)^2 \\ &= \frac{1}{2}h(N,N)^2 - (h \times h)(N,N). \end{aligned}$$

The result now follows from Lemma 40. □

We now record some computations that will prove useful in evaluating the above expressions. For any (not necessarily tangential) vector field Z defined on Σ, we let $\delta_\Sigma Z$ denote the divergence of the tangential part of Z, computed with respect to the induced metric on Σ. The following result relates this to the full divergence, δZ.

Lemma 43. *For any vector field Z on Σ we have*

$$\delta Z = \delta_\Sigma Z + \langle Z, N\rangle H - \langle \nabla_N Z, N\rangle.$$

Note that setting $Z = N$ immediately yields the formula $H = \delta N$.

Proof. In an adapted orthonormal frame we write.

$$\delta Z = -\sum_\alpha \langle \nabla_\alpha Z, E_\alpha\rangle - \langle \nabla_N Z, N\rangle.$$

To evaluate the first term we write $Z = Z_\Sigma + Z_N$, where $Z_N := \langle Z, N\rangle N$ and $Z_\Sigma := Z - Z_N$. For the tangential part we have $\nabla_\alpha Z_\Sigma = \nabla^\Sigma_\alpha Z_\Sigma + A(E_\alpha, Z_\Sigma)N$, hence

$$\begin{aligned} -\sum_\alpha \langle \nabla_\alpha Z_\Sigma, E_\alpha\rangle &= -\sum_\alpha \langle \nabla^\Sigma_\alpha Z_\Sigma, E_\alpha\rangle \\ &= \delta_\Sigma Z. \end{aligned}$$

Then

$$\begin{aligned} -\sum_\alpha \langle \nabla_\alpha Z_N, E_\alpha\rangle &= -\langle Z, N\rangle \sum_\alpha \langle \nabla_\alpha N, E_\alpha\rangle \\ &= \langle Z, N\rangle H \end{aligned}$$

by Lemma 28. ☐

We use this lemma to derive the following useful formulae.

Lemma 44. *If* $h_\Sigma = 0$ *and* $\delta_g h = 0$*, then*

$$\begin{aligned} (\nabla_N h)(N,N) =& \delta_\Sigma(h\cdot N) + h(N,N)H; \\ (\nabla_{h\cdot N} h)(N,N) =& 2h(N,N)\delta_\Sigma(h\cdot N) - \delta_\Sigma[h(N,N)h\cdot N] \\ &+ h(N,N)^2 H + 2\langle (h\times h)_\Sigma, A\rangle; \\ 2(\nabla_N h)(N, h\cdot N) =& \delta_\Sigma[(h\times h)\cdot N] + \langle h\otimes N, \nabla h\rangle \\ &+ (h\times h)(N,N)H - \langle (h\times h)_\Sigma, A\rangle \end{aligned}$$

at $t = 0$*.*

Proof. We proceed by applying Lemma 43 to the vector fields $h\cdot N$, $h(N,N)h\cdot N$ and $(h\times h)\cdot N$ respectively. First letting $Z = h\cdot N$, we find that $\delta Z = \langle \delta h, N\rangle - \langle h, \nabla N\rangle$ and

$$\begin{aligned} \langle \nabla_N Z, N\rangle &= (\nabla_N h)(N,N) + h(N, \nabla_N N) \\ &= (\nabla_N h)(N,N) + \langle h, \nabla N\rangle, \end{aligned}$$

where in the last line we have used the fact that $h_\Sigma = 0$. The result then follows from Lemma 43.

We next consider $Z = h(N,N)h \cdot N$, and observe that

$$\begin{aligned}\delta Z &= h(N,N)\delta(h \cdot N) - \nabla_{h \cdot N}(h(N,N)) \\ &= h(N,N)\langle \delta h, N\rangle - h(N,N)\langle h, \nabla N\rangle - \nabla_{h\cdot N}(h(N,N)).\end{aligned}$$

The last term can be written as $\nabla_{h\cdot N}(h(N,N)) = (\nabla_{h\cdot N} h)(N,N) + 2h(N, \nabla_{h\cdot N} N)$, with

$$\begin{aligned}h(N, \nabla_{h\cdot N} N) &= \sum_\alpha h(E_\alpha, N) h(N, \nabla_\alpha N) + h(N,N)h(N, \nabla_N N) \\ &= -\sum_{\alpha\beta} h(E_\alpha, N)h(E_\beta, N)A(E_\alpha, E_\beta) + h(N,N)h(N,\nabla_N N),\end{aligned}$$

using Lemma 28 in the last line. We recognize that

$$\sum_{\alpha\beta} h(E_\alpha, N)h(E_\beta, N)A(E_\alpha, E_\beta) = \langle (h \times h)_\Sigma, A\rangle$$

and thus obtain

$$\begin{aligned}\delta Z &= h(N,N)\langle \delta h, N\rangle - (\nabla_{h\cdot N} h)(N,N) - 3h(N,N)h(N,\nabla_N N) \\ &\quad + \langle 2(h\times h)_\Sigma + h(N,N)h_\Sigma, A\rangle.\end{aligned}$$

We then compute

$$\begin{aligned}\langle \nabla_N Z, N\rangle &= h(N,N)\langle \nabla_N (h\cdot N), N\rangle + h(N,N)\nabla_N(h(N,N)) \\ &= h(N,N)[2(\nabla_N h)(N,N) + 3h(N,\nabla_N N)]\end{aligned}$$

and apply Lemma 43, together with the formula for $(\nabla_N h)(N,N)$ derived above, to obtain the desired result.

Finally, we let $Z = (h\times h)\cdot N$. Using local coordinates we compute

$$\begin{aligned}\delta Z &= -\nabla^i(h_{ij}h^{jk}N_k) \\ &= -(\nabla^i h_{ij})h^{jk}N_k - h_{ij}(\nabla^i h^{jk})N_k - h_{ij}h^{jk}(\nabla^i N_k) \\ &= \langle \delta h, h\cdot N\rangle - \langle h\otimes N, \nabla h\rangle - \langle h\times h, \nabla N\rangle\end{aligned}$$

and

$$\begin{aligned}\langle \nabla_N Z, N\rangle &= N^i \nabla_N (h_{ij} h^{jk} N_k)\\ &= N^i[(\nabla_N h_{ij}) h^{jk} N_k + h_{ij}(\nabla_N h^{jk}) N_k + h_{ij} h^{jk}(\nabla_N N_k)]\\ &= 2(\nabla_N h)(N, h\cdot N) + (h\times h)(N, \nabla_N N).\end{aligned}$$

We then note that

$$\begin{aligned}\langle h\times h, \nabla N\rangle &= \sum_\alpha (h\times h)(E_\alpha, \nabla_\alpha N) + (h\times h)(N, \nabla_N N)\\ &= -\langle (h\times h)_\Sigma, A\rangle + (h\times h)(N, \nabla_N N)\end{aligned}$$

by Lemma 28. The result now follows from Lemma 43. □

We can now compute the trace term that occurs in the first and second variation of mean curvature.

Lemma 45. *If $h_\Sigma = 0$ and $\delta_g h = 0$, then*

$$2\,\mathrm{tr}_\Sigma \left\langle \frac{\partial}{\partial t}\nabla, N\right\rangle = -\delta_\Sigma(h\cdot N) - h(N,N)H - \nabla_N(\mathrm{tr}\, h)$$

at $t = 0$.

Proof. Using adapted orthonormal coordinates, we have

$$\begin{aligned}2\,\mathrm{tr}_\Sigma \left\langle \frac{\partial}{\partial t}\nabla, N\right\rangle &= \sum_\alpha 2(\nabla_\alpha h)(E_\alpha, N) - (\nabla_N h)(E_\alpha, E_\alpha)\\ &= -2\langle \delta h, N\rangle - 2(\nabla_N h)(N,N) + (\nabla_N h)(N,N) - \nabla_N(\mathrm{tr}\, h)\\ &= -2\langle \delta h, N\rangle - (\nabla_N h)(N,N) - \nabla_N(\mathrm{tr}\, h).\end{aligned}$$

We use Lemma 44 to compute $(\nabla_N h)(N,N)$ and the proof follows. □

Now the proof of Proposition 39 follows from Proposition 42 and Lemma 45.

We record one additional consequence of Lemma 44 that will be useful in the following section.

Lemma 46. *If* $h_\Sigma = 0$ *and* $\delta_g h = 0$*, then*

$$\begin{aligned}\langle h \otimes N, \nabla h\rangle - 2h(N, \nabla \operatorname{tr} h) - \nabla_N |h|^2 =& [2h(N,N)^2 - 3(h \times h)(N,N)]H \\ &- 2h(N,N)\,[\delta_\Sigma(h \cdot N) + \nabla_N(\operatorname{tr} h)] \\ &- \langle (h \times h)_\Sigma, A\rangle - 3\delta_\Sigma[(h \times h) \cdot N]\end{aligned}$$

at $t = 0$.

Proof. We first use the fact that $h_\Sigma = 0$ to obtain

$$\begin{aligned}\langle h \otimes N, \nabla h\rangle =& \sum_\alpha [h(E_\alpha, N)(\nabla_\alpha h)(N,N) + h(E_\alpha, N)(\nabla_N h)(E_\alpha, N)] \\ &+ h(N,N)(\nabla_N h)(N,N) \\ =& (\nabla_{h\cdot N} h)(N,N) + (\nabla_N h)(N, h \cdot N) - h(N,N)(\nabla_N h)(N,N)\end{aligned}$$

and

$$\begin{aligned}\langle h, \nabla_N h\rangle &= 2\sum_\alpha h(E_\alpha, N)(\nabla_N h)(E_\alpha, N) + h(N,N)(\nabla_N h)(N,N) \\ &= 2(\nabla_N h)(N, h \cdot N) - h(N,N)(\nabla_N h)(N,N).\end{aligned}$$

We then combine this with Lemma 44 to write

$$\begin{aligned}-2\langle h \otimes N, \nabla h\rangle - \nabla_N |h|^2 =& -2(\nabla_{h\cdot N} h)(N,N) - 6(\nabla_N h)(N, h \cdot N) \\ &+ 4h(N,N)(\nabla_N h)(N,N) \\ =& -3\langle h \otimes N, \nabla h\rangle + 2h(N,N)^2 H - 3(h \times h)(N,N)H \\ &+ \delta_\Sigma\,[2h(N,N)h \cdot N - 3(h \times h) \cdot N] - \langle (h \times h)_\Sigma, A\rangle.\end{aligned}$$

Finally, we observe that $h(N,N) = \operatorname{tr} h$ on Σ, so

$$\begin{aligned}\delta_\Sigma[h(N,N)h \cdot N] =& \delta_\Sigma[(\operatorname{tr} h)h \cdot N] \\ =& h(N,N)\delta_\Sigma(h \cdot N) - h(N, \nabla^\Sigma \operatorname{tr} h) \\ =& h(N,N)\delta_\Sigma(h \cdot N) - h(N, \nabla \operatorname{tr} h) + h(N,N)\nabla_N(\operatorname{tr} h)\end{aligned}$$

The desired formula follows upon combining the last two results. □

2.4 Proof of Theorem 30

We will prove Theorem 30 by computing the first and second derivatives of the functional $\mathcal{F}_\Omega$, then appealing to the Morse–Palais lemma. The proof of this lemma is standard, and can be found in Lang (1995).

Lemma 47. *Let B be a Banach space, U an open neighborhood of $x_0 \in B$, and $F : U \to \mathbb{R}$ a function of class C^{k+2} with $k \geq 1$. Suppose x_0 is a nondegenerate critical point of F. Then there exists a neighborhood V of $0 \in B$ and a C^k diffeomorphism $\xi : V \to \xi(V) \subset U$ such that $\xi(0) = x_0$, and*

$$F(\xi(y)) = F(x_0) + \frac{1}{2} D^2 F(x_0)(y, y)$$

for all $y \in V$. Moreover, ξ satisfies $D\xi(0) = Id$.

We would like to apply this lemma to $F(h) := \mathcal{F}_\Omega(\bar{g} + h)$ on the space

$$B := \{h \in W^{2,p} : \bar{\delta} h = 0 \text{ and } h_\Sigma = 0\}.$$

Once we establish that 0 is a nondegenerate critical point of F, it will follow that

$$\mathcal{F}_\Omega(\bar{g} + \xi(h)) = \frac{1}{2} D^2 \mathcal{F}_\Omega(\bar{g})(h, h)$$

for all h in some $W^{2,p}$ neighborhood $V \subset B$ containing 0. We then use the following slice theorem to extend this formula to all Riemannian metrics sufficiently close to $\bar{g}$. This is a minor generalization of a result of Brendle and Marques (2010), following the classic work of Ebin (1968).

Proposition 48. *Fix $p > n$. If $\|g - \bar{g}\|_{W^{2,p}(\Omega,\bar{g})}$ is sufficiently small, then there exists a diffeomorphism $\varphi : \Omega \to \Omega$ such that $\varphi|_\Sigma = id$, and $h := \xi^{-1}(\varphi^* g - \bar{g})$ has $\bar{\delta} h = 0$. Moreover,*

$$\|h\|_{W^{2,p}(\Omega,\bar{g})} \leq C \|g - \bar{g}\|_{W^{2,p}(\Omega,\bar{g})}$$

for some constant $C = C(\Omega, p)$.

From this it follows that

$$\mathcal{F}_\Omega(\varphi^* g) = \frac{1}{2} D^2 \mathcal{F}_\Omega(\bar{g})(h,h);$$

so the proof is completed by an explicit computation of $D^2\mathcal{F}_\Omega(\bar{g})(h,h)$ for divergence-free h.

We thus proceed by computing the first and second derivatives of $\mathcal{F}_\Omega$. These computations will demonstrate that $\bar{g}$ is a critical point of $\mathcal{F}_\Omega$, and additionally provide the desired formula for $D^2\mathcal{F}_\Omega(\bar{g})(h,h)$. The chapter then concludes with the proof of Proposition 48.

We begin by recording some useful integration by parts formulae for $\bar{g}$. Though the results are elementary, we write them out in full to make explicit the chosen sign conventions for N, δ and Δ. We recall that f satisfies the equation $\operatorname{Hess} f = -f\bar{g}$, hence $-\Delta f = nf$, on Ω.

Lemma 49. *Let h be a symmetric $(0,2)$-tensor, and u a smooth function, on Ω. Then*

$$\int_\Omega f\delta^2 h = -\int_\Omega f(\operatorname{tr} h) + \int_\Sigma \left[f\langle \delta h, N\rangle + h(\nabla f, N)\right],$$

$$\int_\Omega f\Delta u = -n\int_\Omega uf + \int_\Sigma \left[u\nabla_N f - f\nabla_N u\right].$$

If $\delta h = 0$ then

$$\int_\Omega f\langle h, \operatorname{Hess} u\rangle = -\int_\Omega uf(\operatorname{tr} h) + \int_\Sigma \left[uh(\nabla f, N) - fh(\nabla u, N)\right].$$

Proof. To prove the first statement, we let $X = f\delta h + h(\nabla f, \cdot)$. Then

$$\begin{aligned}\delta X &= f\delta^2 h - \langle df, \delta h\rangle + \langle \delta h, df\rangle - \langle h, \operatorname{Hess} f\rangle \\ &= f\delta^2 h - \langle h, \operatorname{Hess} f\rangle.\end{aligned}$$

The proof is completed by the observation that $\langle h, \operatorname{Hess} f\rangle = -f(\operatorname{tr} h)$.

For the second claim, we let $X = udf - fdu$. Using the fact that $\Delta = -\delta d$, we compute

$$\delta X = f\Delta u - u\Delta f.$$

Since, $\nabla f = -nf$, the result follows.

For the final claim, we let $X = h(u\nabla f - f\nabla u, \cdot)$ and compute

$$\delta X = \langle \delta h, u\nabla f - f\nabla u\rangle + f\langle h, \operatorname{Hess} u\rangle - u\langle h, \operatorname{Hess} f\rangle.$$

As above, we note that $\langle h, \operatorname{Hess} f\rangle = -f(\operatorname{tr} h)$ and the result follows. □

Having established these results, we proceed to differentiate the total scalar curvature functional.

Proposition 50. *Suppose $h_\Sigma = 0$ and $\bar{\delta} h = 0$. Then $D\mathcal{F}_\Omega(\bar{g})(h) = 0$ and*

$$\begin{aligned} D^2\mathcal{F}_\Omega(\bar{g})(h,h) = &- \int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}(d\operatorname{tr} h)^2 + |h|^2 + (\operatorname{tr} h)^2\right) \\ &+ \int_\Sigma \left[-2h(N,N)h(\nabla^\Sigma f, N) + \big(h(N,N)^2 + (h\times h)(N,N)\big)\nabla_N f\right] \\ &+ \int_\Sigma f\left[\frac{1}{2}h(N,N)^2 H - (h\times h)(N,N)H - \langle (h\times h)_\Sigma, A\rangle\right]. \end{aligned}$$

Proof. We let $g_t = \bar{g} + th$. Since $|\bar{N}|_{\bar{g}}^{-1} = 1$, we have

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_{t=0} |\bar{N}|_{g_t}^{-1}\left(H_{g_t} - \bar{H}\right) &= \left.\frac{\partial H}{\partial t}\right|_{t=0} \\ &= -\frac{1}{2}\delta_\Sigma(h\cdot N) - \frac{1}{2}\nabla_N(\operatorname{tr} h), \end{aligned}$$

by Proposition 39, hence

$$D\mathcal{F}_\Omega(\bar{g})(h) = \int_\Omega f\,\mathrm{L}_{\bar{g}}\,h - \int_\Sigma f[\delta_\Sigma(h\cdot N) + \nabla_N(\operatorname{tr} h)].$$

Then lemma 49 implies

$$\int_\Omega f\,\mathrm{L}_{\bar{g}}\,h = \int_\Sigma [f\nabla_N(\operatorname{tr} h) + h(\nabla f, N) - h(N,N)\nabla_N f].$$

We observe that

$$-f\delta_\Sigma(h\cdot N) + h(\nabla f, N) - h(N,N)\nabla_N f = -\delta_\Sigma[f(h\cdot N)]$$

integrates to zero by Stokes' theorem, and the first result follows.

To compute the second derivative we first observe the following consequences of Lemma 49:

$$\int_\Omega f\delta^2(h\times h) = -\int_\Omega f|h|^2 + \int_\Sigma [(h\times h)(\nabla f, N) - f\langle h\otimes N, \nabla h\rangle];$$

$$\int_\Omega f\Delta|h|^2 = -n\int_\Omega f|h|^2 + \int_\Sigma \left[|h|^2\nabla_N f - f\nabla_N|h|^2\right];$$

and

$$\int_\Omega f\langle h, \operatorname{Hess}(\operatorname{tr} h)\rangle = -\int_\Omega f(\operatorname{tr} h)^2 + \int_\Sigma \left[(\operatorname{tr} h)h(\nabla f, N) - fh(\nabla trh, N)\right].$$

In the first equation we have used the fact that $\langle\delta(h\times h), N\rangle = -\langle h\otimes N, \nabla h\rangle$ when $\delta h = 0$. We next recall from Equation (2.2) that

$$\begin{aligned} \mathrm{Q}_g(h) = &-\frac{1}{2}|\nabla h|^2 + \langle h\times h, \operatorname{Ric}\rangle - \frac{1}{2}(d\operatorname{tr} h)^2 + 2\langle h, \operatorname{Hess}(\operatorname{tr} h)\rangle \\ &+ \Delta|h|^2 - \delta^2(h\times h) + R_{ijkl}h^{il}h^{jk} \end{aligned}$$

The spherical metric has curvature tensor $R_{ijkl} = g_{il}g_{jk} - g_{jl}g_{ik}$, hence $R_{jk} = (n-1)g_{jk}$. This implies $\langle h\times h, \operatorname{Ric}\rangle = (n-1)|h|^2$ and $R_{ijkl}h^{il}h^{jk} = (\operatorname{tr} h)^2 - |h|^2$. We thus obtain

$$\begin{aligned} \int_\Omega fQ_{\bar{g}}(h) = &-\int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}(d\operatorname{tr} h)^2 + |h|^2 + (\operatorname{tr} h)^2\right) \\ &+ \int_\Sigma \left[2(\operatorname{tr} h)h(\nabla f, N) + |h|^2\nabla_N f - (h\times h)(\nabla f, N)\right] \\ &+ \int_\Sigma f\left[\langle h\otimes N, \nabla h\rangle - 2h(\nabla\operatorname{tr} h, N) - \nabla_N|h|^2\right]. \end{aligned} \tag{2.3}$$

To differentiate the boundary terms we write $|\bar{N}|_{g_t}^{-1} = g_t(\bar{N}, \bar{N})^{-1/2}$, from which it follows that

$$\frac{\partial}{\partial t}|\bar{N}|_{g_t}^{-1} = -\frac{1}{2}g_t(\bar{N}, \bar{N})^{-3/2}h(\bar{N}, \bar{N}).$$

Therefore

$$\begin{aligned}\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} |\bar{N}|_{g_t}^{-1}\left(H_{g_t} - \bar{H}\right) &= -\,h(N,N)\left.\frac{\partial H}{\partial t}\right|_{t=0} + \left.\frac{\partial^2 H}{\partial t^2}\right|_{t=0} \\ &= \left[(h\times h)(N,N) - \frac{3}{4}h(N,N)^2\right]H \\ &\quad + h(N,N)\left[\delta_\Sigma(h\cdot N) + \nabla_N(\operatorname{tr} h)\right]\end{aligned}$$

by Proposition 39. By combining this with Equation (2.3) and applying Lemma 46, we see that the combined boundary terms occurring in $D^2\mathcal{F}_\Omega$ are given by

$$\begin{aligned}&2h(N,N)h(\nabla f, N) + |h|^2\nabla_N f - (h\times h)(\nabla f, N) \\ &+ f[2h(N,N)^2 - 3(h\times h)(N,N)]H - 2fh(N,N)\left[\delta_\Sigma(h\cdot N) + \nabla_N(\operatorname{tr} h)\right] \\ &- f\langle (h\times h)_\Sigma, A\rangle - 3f\delta_\Sigma[(h\times h)\cdot N] \\ &+ 2f\left[(h\times h)(N,N) - \frac{3}{4}h(N,N)^2\right]H + 2fh(N,N)\left[\delta_\Sigma(h\cdot N) + \nabla_N(\operatorname{tr} h)\right].\end{aligned}$$

The proof follows from the above formula, together with the decomposition $\nabla f = \nabla^\Sigma f + (\nabla_N f)N$, and the observations that

$$\delta_\Sigma[f(h\times h)\cdot N] = f\delta_\Sigma[(h\times h)\cdot N] - (h\times h)(\nabla^\Sigma f, N)$$

and

$$2h(N,N)^2 + |h|^2 - (h\times h)(N,N) = h(N,N)^2 + (h\times h)(N,N)$$

on Σ, because $h_\Sigma = 0$. □

This immediately implies 0 is a critical point of F; it is nondegenerate by assumption on Ω, so the Morse–Palais lemma applies. We finish with the proof of the slice theorem.

Proof. (of Proposition 48) We closely follow the approach and notation of Brendle and Marques (2010). We let $\mathcal{M}$ denote the space of $W^{2,p}$ Riemannian metrics on Ω, and $\mathcal{D}$ the space of $W^{3,p}$ diffeomorphisms that fix Σ. The tangent spaces of $\mathcal{M}$ and $\mathcal{D}$ are given by $\mathcal{S}$, the space of $W^{2,p}$ symmetric $(0,2)$-tensors, and $\mathcal{X}$, the space of vector fields in $W^{3,p} \cap W_0^{1,p}$. Additionally, we let $\mathcal{Y}$ denote the space of $W^{1,p}$ vector fields on Ω.

We then define a map

$$\begin{aligned} F : \mathcal{D} \times \mathcal{M} &\to \mathcal{Y} \\ (\varphi, g) &\mapsto \bar{\delta}(\xi^{-1}(\varphi^* g - \bar{g})). \end{aligned}$$

Differentiating this with respect to φ and evaluating at $(id, \bar{g})$, we find

$$\begin{aligned} DF(\bar{g})(X) &= \bar{\delta}\left(D\xi^{-1}(0)(\mathcal{L}_X \bar{g})\right) \\ &= \bar{\delta}\left(\mathcal{L}_X \bar{g}\right), \end{aligned}$$

where in the last line we have used the fact that $\xi(0) = 0$ and $D\xi(0) = \mathrm{Id}$.

We next show that $DF(\bar{g}) : \mathcal{X} \to \mathcal{Y}$ is an isomorphism; after this is established the result will follow from the implicit function theorem and the fact that $F(id, \bar{g}) = 0$.

It was observed by Brendle and Marques (see also Berger and Ebin (1969)) that

$$\mathcal{S} = \{\mathcal{L}_X \bar{g} : X \in \mathcal{X}\} \oplus \{h \in \mathcal{S} : \bar{\delta} h = 0\},$$

whence $\ker DF(\bar{g}) = \{X \in \mathcal{X} : \mathcal{L}_X \bar{g} = 0\}$. Each Killing vector field on $\mathbb{S}^n$ is tangent to a rotation $M \in SO(n+1)$. If M is not the identity, it can have at most $n-1$ eigenvalues equal to 1, hence the set of points in $\mathbb{R}^{n+1}$ fixed by M is a plane through the origin of dimension at most $n-1$. This plane will intersect $\mathbb{S}^n$ in a submanifold of dimension at most $n-2$, hence there is no nontrivial Killing vector field on $\mathbb{S}^n$ that vanishes on a hypersurface. It follows that $\ker DF(\bar{g})$ is trivial.

We complete the proof by demonstrating surjectivity of $DF(\bar{g}) = 2\bar{\delta}\bar{\delta}^*$. This is an injective elliptic operator with trivial kernel, so we have that the bilinear form

$$B(X, Y) := \int_\Omega \left\langle \bar{\delta}^* X, \bar{\delta}^* Y \right\rangle$$

is strictly coercive over $W_0^{1,2}$. Now let $Y \in \mathcal{Y}$. Since $p > n > 2$, we have $Y \in L^2$. Then the Lax–Milgram lemma (see Folland (1995) for details) implies the existence of a weak solution $X \in W_0^{1,2}$ to the problem

$$2\bar{\delta}\bar{\delta}^* X = Y.$$

Since $Y \in W^{1,p}$, elliptic regularity (Theorem 9.19 of Gilbarg and Trudinger (1983)) implies $X \in W^{3,p}$, and the proof follows. □

3

Local rigidity theorems

In this Chapter we prove the local rigidity theorems stated in the introduction. To prove a rigidity theorem on a domain $\Omega \subset \mathbb{S}^n$, we first show that Ω is nondegenerate (according to the definition of Chapter 2). We can then apply Theorem 30 to any metric g suitably close to $\bar{g}$, to find a diffeomorphism φ such that

$$2\mathcal{F}_\Omega(\varphi^* g) = -\int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}|d\,\mathrm{tr}\,h|^2 + |h|^2 + (\mathrm{tr}\,h)^2\right) + \text{boundary terms.} \tag{3.1}$$

If g satisfies the inequalities $R_g \geq \bar{R}$ and $H_g \geq \bar{H}$, then $\varphi^* g$ will also, hence $\mathcal{F}_\Omega(\varphi^* g) \geq 0$. We next show that the right-hand side of Equation (3.1) is negative definite; since the left-hand side is nonnegative, we conclude that $h \equiv 0$. Recalling the proof of Proposition 48, we have $h = \xi^{-1}(\varphi^* g - \bar{g})$ and $\xi(0) = 0$. This implies $\varphi^* g - \bar{g} = 0$, completing the proof.

We therefore must: 1) establish the nondegeneracy of Ω; and 2) show that the right-hand side of Equation (3.1) is negative definite for any domain Ω on which we wish to prove a scalar curvature rigidity theorem. Since the right-hand side of Equation (3.1) is precisely the second derivative $D^2\mathcal{F}_\Omega(\bar{g})(h,h)$, the nondegeneracy of Ω will immediately follow once we show that $D^2\mathcal{F}_\Omega$ is negative definite.

Since the interior terms in Equation (3.1) are all negative, the only positive contributions come from the boundary. In the case of a geodesic ball $\{f \geq c\}$, it was observed by Brendle and Marques (2010) that the boundary terms are nonpositive provided

$$c \geq \frac{4}{n+3}.$$

We will see that it is possible to obtain a stronger result by utilizing the negative interior terms, since we only need the *sum* of the interior and boundary terms to be nonpositive. More specifically, we show in the next section that the boundary terms can be controlled by the interior terms; when the control is good enough (as will be the case when Ω is sufficiently small) it will follow that $D^2\mathcal{F}_\Omega$ is negative definite, as desired.

We next generalize these results to domains in the hemisphere that are not geodesic balls. Here we require additional control on the extrinsic geometry of the boundary (which is no longer totally umbilic as it was for a geodesic ball). In particular, we prove a local rigidity theorem for domains $\Omega \subset \{f \geq c\}$ having "sufficiently large" mean curvature (with respect to $\bar{g}$), where the precise notion of sufficiently large depends on c and the smallest eigenvalue of the second fundamental form (again computed with respect to $\bar{g}$).

In the final section we take a more systematic approach to the problem of finding the largest domain on which the local rigidity theorem is true. We prove some existence results for a related variational problem, and make a conjecture relating degenerate geodesic balls to the index of a symmetric elliptic operator on Ω.

3.1 Weighted boundary estimates for geodesic balls

We begin by estimating the $h(N,N)^2$ boundary term that arises in Corollary 31. We are interested in finding an estimate of the form

$$\int_\Sigma h(N,N)^2 \leq K \int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}|d\,\mathrm{tr}\,h|^2 + |h|^2 + (\mathrm{tr}\,h)^2\right)$$

for $K > 0$. In the next section it will be shown that such an estimate yields a rigidity statement for Ω, provided K is sufficiently small. We are thus interested in finding, on a given domain Ω, the smallest constant K such that the above inequality holds for all symmetric, divergence-free tensors h. In this section we establish such an inequality for any geodesic ball $\{f \geq c\}$ with $c > 0$, from which the proof of Theorem 22 will readily follow.

Proposition 51. *Let h be a divergence-free, symmetric $(0,2)$-tensor on $\Omega = \{f \geq c\}$. Then*

$$\sqrt{1-c^2}\int_\Sigma (\mathrm{tr}\,h)h(N,N) \leq \int_\Omega \left(\frac{w}{2}\sqrt{1-f^2}|h|^2 + f(\mathrm{tr}\,h)^2 + \frac{1}{2w}\sqrt{1-f^2}|d\,\mathrm{tr}\,h|^2\right)$$

for any positive function $w \in C^0(\Omega)$.

Proof. Consider the one-form $X = (\mathrm{tr}\,h)h \cdot \nabla f$. On Σ we have $\nabla f = \sqrt{1-c^2}N$, hence

$$\langle X, N\rangle = \sqrt{1-c^2}(\mathrm{tr}\,h)h(N,N).$$

Using the fact that $\delta h = 0$, we compute

$$\begin{aligned}\delta X &= -h(\nabla f, \nabla\,\mathrm{tr}\,h) - (\mathrm{tr}\,h)\langle h, \mathrm{Hess}\,f\rangle \\ &= f(\mathrm{tr}\,h)^2 - h(\nabla f, \nabla\,\mathrm{tr}\,h),\end{aligned}$$

where we have recalled that $\mathrm{Hess}\,f = -fg$. We apply the Cauchy–Schwartz inequality to the second term to obtain

$$\begin{aligned}-h(\nabla f, \nabla\,\mathrm{tr}\,h) &\leq |\nabla f||h||d\,\mathrm{tr}\,h| \\ &= \sqrt{1-f^2}|h||d\,\mathrm{tr}\,h|.\end{aligned}$$

Finally, we apply the arithmetic-geometric mean inequality, $2ab \le \lambda a^2 + (1/\lambda)b^2$, at each point $x \in \Omega$, with $a = |h|(x)$, $b = |d\,\mathrm{tr}\,h|(x)$, and $\lambda = w(x)$. The result follows from the divergence theorem. □

If the tangential part of h vanishes on Σ, then $\mathrm{tr}\,h = h(N,N)$, so the above inequality, with $w = \sqrt{2}$, implies

$$\sqrt{1-c^2}\int_\Sigma h(N,N)^2 \le \int_\Omega \left(\sqrt{\frac{1-c^2}{2}}\left(|h|^2 + \frac{1}{2}|d\,\mathrm{tr}\,h|^2\right) + f(\mathrm{tr}\,h)^2\right). \tag{3.2}$$

3.2 Local rigidity for geodesic balls

We are now in a position to complete the proof of Theorem 22. Recall that g is a metric on Ω with $g_\Sigma = \bar{g}_\Sigma$, $R_g \ge \bar{R}$ and $H_g \ge \bar{H}$. Assuming that g is sufficiently close to $\bar{g}$ in $W^{2,p}$, we have from Corollary 31 that

$$\begin{aligned} 2\mathcal{F}_\Omega(\varphi^* g) = &-\int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}(d\,\mathrm{tr}\,h)^2 + |h|^2 + (\mathrm{tr}\,h)^2\right) \\ &+ \left(\sqrt{1-c^2} - \frac{nc^2}{\sqrt{1-c^2}}\right)\int_\Sigma (h \times h)(N,N) \\ &+ \left(\sqrt{1-c^2} + \frac{(n+1)c^2}{2\sqrt{1-c^2}}\right)\int_\Sigma h(N,N)^2. \end{aligned}$$

for some diffeomorphism φ with $\varphi|_\Sigma = id$, and some symmetric, divergence-free tensor h with $h_\Sigma = 0$. Diffeomorphism invariance of the scalar curvature immediately implies $R_{\varphi^* g} = \varphi^* R_g \ge \bar{R}$ at each point in Ω. The following elementary lemma proves that $H_{\varphi^* g} \ge \bar{H}$.

Lemma 52. *Let $\Sigma \subset M$ be a smoothly embedded submanifold, and $\varphi : M \to M$ a diffeomorphism such that $\varphi(\Sigma) = \Sigma$. Then $H_{\varphi^* g} = \varphi^* H_g$ on Σ for any Riemannian metric g on M.*

Proof. For convenience, we let $\tilde{g} = \varphi^* g$. Fix $p \in \Sigma$, and let $E_1, \ldots, E_n = N_g$ be an adapted g-orthonormal basis in a neighborhood of p. Then the vector fields

$\tilde{E}_i = (\varphi^{-1})_* E_i$ form a $\tilde{g}$-orthonormal basis in a neighborhood of $\varphi^{-1}(p)$. Since φ maps Σ into Σ, we see that $\tilde{E}_1, \ldots, \tilde{E}_{n-1}$ are tangential vector fields, hence $\tilde{E}_n = N_{\tilde{g}}$, the $\tilde{g}$-unit normal. We compute

$$\begin{aligned} H_{\tilde{g}}(\varphi^{-1}(p)) &= -\sum_{i=1}^{n} \tilde{g}\left(\tilde{\nabla}_{\tilde{E}_i}\tilde{N}, \tilde{E}_i\right)(\varphi^{-1}(p)) \\ &= -\sum_{i=1}^{n} g\left(\varphi_*\left(\tilde{\nabla}_{\tilde{E}_i}\tilde{N}\right), \varphi_*\left(\tilde{E}_i\right)\right)(p) \\ &= -\sum_{i=1}^{n} g\left(\nabla_{E_i} N, E_i\right)(p) \\ &= H_g(p) \end{aligned}$$

and the result follows. □

It follows immediately that $\mathcal{F}_\Omega(\varphi^* g) \geq 0$, hence

$$\begin{aligned} 0 \leq & D^2\mathcal{F}_\Omega(\bar{g})(h,h) \\ = & -\int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}|d\operatorname{tr} h|^2 + |h|^2 + (\operatorname{tr} h)^2\right) \\ & + \left(\sqrt{1-c^2} - \frac{nc^2}{\sqrt{1-c^2}}\right)\int_\Sigma (h\times h)(N,N) \\ & + \left(\sqrt{1-c^2} + \frac{(n+1)c^2}{2\sqrt{1-c^2}}\right)\int_\Sigma h(N,N)^2. \end{aligned}$$

Since we are assuming that $c^2 \geq \frac{1}{n+1}$, and $h(N,N)^2 \leq (h\times h)(N,N)$ by definition, the first boundary term satisfies

$$\left(\sqrt{1-c^2} - \frac{nc^2}{\sqrt{1-c^2}}\right)\int_\Sigma (h\times h)(N,N) \leq \left(\sqrt{1-c^2} - \frac{nc^2}{\sqrt{1-c^2}}\right)\int_\Sigma h(N,N)^2$$

Combining this with Equation (3.2), we find that

$$\begin{aligned} D^2\mathcal{F}_\Omega(\bar{g})(h,h) \leq & -\int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}|d\operatorname{tr} h|^2 + |h|^2 + (\operatorname{tr} h)^2\right) \\ & + \left(2\sqrt{1-c^2} - \frac{(n-1)c^2}{2\sqrt{1-c^2}}\right)\int_\Omega \left[\frac{1}{\sqrt{2}}\left(|h|^2 + \frac{1}{2}|d\operatorname{tr} h|^2\right) + \frac{f}{\sqrt{1-c^2}}(\operatorname{tr} h)^2\right]. \end{aligned}$$

If c is chosen to satisfy the inequalities

$$c + \frac{1}{\sqrt{2}} \left(\frac{(n-1)c^2}{2\sqrt{1-c^2}} - 2\sqrt{1-c^2} \right) \geq 0 \tag{3.3}$$

$$1 + \frac{1}{\sqrt{1-c^2}} \left(\frac{(n-1)c^2}{2\sqrt{1-c^2}} - 2\sqrt{1-c^2} \right) \geq 0, \tag{3.4}$$

then

$$D^2\mathcal{F}_\Omega(\bar{g})(h,h) \leq -\frac{1}{2} \int_\Omega \left[f|\nabla h|^2 + 2(f-c)|h|^2 \right].$$

This is nonpositive for all h because $f \geq c$ in Ω. Moreover, if $D^2\mathcal{F}_\Omega(\bar{g})(h,h) = 0$ then $\nabla h = 0$ and $(f-c)|h|^2 = 0$ in Ω, hence $h \equiv 0$. Thus $D^2\mathcal{F}_\Omega(\bar{g})$ is negative definite.

The following lemma completes the proof of Theorem 22.

Lemma 53. *Inequalities (3.3) and (3.4) are satisfied when*

$$c^2 \geq \begin{cases} \frac{2}{n+1} & \text{if } n \leq 5, \\ 4\left(\frac{4+n-\sqrt{2n-1}}{n^2+6n+17}\right) & \text{if } n > 5. \end{cases}$$

Proof. We start by rearranging (3.3) into the equivalent form

$$4 - (n+3)c^2 \leq 2\sqrt{2}c\sqrt{1-c^2}.$$

It is easy to verify that the function $u(x) = 2\sqrt{2}x\sqrt{1-x^2} + (n+3)x^2 - 4$ has $u(0) < 0$, and $u(x) > 0$ for $x \geq \frac{2}{\sqrt{n+3}}$. We claim that u has precisely one root, x_0, in the interval $[0,1]$, hence $u(x) \geq 0$ for all $x \geq x_0$. To see this, we square the equation $4 - (n+3)x^2 = 2\sqrt{2}x\sqrt{1-x^2}$ to find the following quadratic equation (in $z = x^2$):

$$(n^2 + 6n + 17)z^2 - 8(n+4)z + 16 = 0.$$

After some simplification, we find the roots

$$z_\pm = 4\left(\frac{n+4 \pm \sqrt{2n-1}}{n^2+6n+17}\right).$$

A simple computation shows that $z_+ \geq 4/(n+3)$, hence $u(\sqrt{z_+}) > 0$. It follows that the desired root is $x_0 = \sqrt{z_-}$.

Similarly, we find that (3.4) is satisfied precisely when $c^2 \geq \frac{2}{n+1}$. A straightforward computation shows that

$$4\left(\frac{4+n-\sqrt{2n-1}}{n^2+6n+17}\right) \geq \frac{2}{n+1},$$

exactly when $n \geq 5$ (with equality for $n = 5$), and the proof follows. $\square$

3.3 Rigidity for arbitrary domains in the hemisphere

We next come to the proof of Theorem 23, for nonspherical domains $\Omega \subset \mathbb{S}^n_+$. We recall that $\Omega \subset \{f \geq c\}$ and

$$\bar{H} \geq \frac{1}{2c}\left(-\lambda + 5\sqrt{1-c^2} + \sqrt{\lambda^2 + 6\lambda\sqrt{1-c^2} + 17(1-c^2)}\right) \tag{3.5}$$

on Σ, where λ is some function satisfying $\bar{A} \geq \lambda \bar{g}$. We emphasize, as above, that these are bounds on the extrinsic geometry of Σ computed with respect to $\bar{g}$, which ensure that the local rigidity theorem is true for any metric g on Ω having $R_g \geq \bar{R}$ and $H_g \geq \bar{H}$.

Proof. (of Theorem 23) Following the proof of Theorem 22, it suffices to prove that $D^2\mathcal{F}_\Omega(\bar{g})$ is negative definite on the space of divergence-free tensors with $h_\Sigma = 0$. We have from Theorem 30 that

$$\begin{aligned} D^2\mathcal{F}_\Omega(\bar{g})(h,h) = &-\int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}(d\,\mathrm{tr}\,h)^2 + |h|^2 + (\mathrm{tr}\,h)^2\right) \\ &+ \int_\Sigma \left[-2h(N,N)h(\nabla^\Sigma f, N) + \big(h(N,N)^2 + (h\times h)(N,N)\big)\nabla_N f\right] \\ &+ \int_\Sigma f\left[\frac{1}{2}h(N,N)^2 H - (h\times h)(N,N)H - \langle (h\times h)_\Sigma, A\rangle\right]. \end{aligned}$$

For convenience we set $X = (h \cdot N)_\Sigma$, so that $(h \times h)(N, N) = h(N, N)^2 + |X|^2$. We then have

$$\begin{aligned} -2h(N,N)h(\nabla^\Sigma f, N) &\leq 2h(N,N)|X||\nabla^\Sigma f| \\ &\leq a\sqrt{1-c^2}h(N,N)^2 + \frac{\sqrt{1-c^2}}{a}|X|^2 \end{aligned}$$

for any positive function a, by the arithmetic-geometric mean inequality. For the remaining boundary terms we have

$$\left(h(N,N)^2 + (h \times h)(N,N)\right) \nabla_N f \leq \left(2h(N,N)^2 + |X|^2\right)\sqrt{1-c^2}$$

and

$$\begin{aligned} f\left[\frac{1}{2}h(N,N)^2 H - (h \times h)(N,N)H\right] &= -fH\left[\frac{1}{2}h(N,N)^2 + |X|^2\right] \\ &\leq -cH\left[\frac{1}{2}h(N,N)^2 + |X|^2\right] \end{aligned}$$

because $H \geq 0$. (This is the case because $H \geq \bar{H}$, and (3.5) implies $\bar{H} \geq 0$.) Finally, we observe that

$$\begin{aligned} \langle (h \times h)_\Sigma, A \rangle &= A(X, X) \\ &\geq \lambda |X|^2. \end{aligned}$$

It follows that the boundary terms are bounded above by

$$\left[a\sqrt{1-c^2} + 2\sqrt{1-c^2} - \frac{c}{2}H\right] h(N,N)^2 + \left[\frac{\sqrt{1-c^2}}{a} + \sqrt{1-c^2} - cH - \lambda\right]|X|^2;$$

this expression will be nonpositive provided the inequalities

$$cH \geq (2a+4)\sqrt{1-c^2} \tag{3.6}$$

and

$$cH \geq (1 + a^{-1})\sqrt{1-c^2} - \lambda \tag{3.7}$$

are both satisfied. Since a is arbitrary, we should choose it to minimize

$$\max\left((2a+4)\sqrt{1-c^2}, (1+a^{-1})\sqrt{1-c^2}-\lambda\right);$$

this yields the weakest possible lower bound on H that ensures inequalities (3.6) and (3.7) are both satisfied. Since $(2a+4)\sqrt{1-c^2}$ is an increasing function of a and $(1+a^{-1})\sqrt{1-c^2}-\lambda$ is decreasing, the maximum of these two quantities will be minimized when they are equal:

$$(2a+4)\sqrt{1-c^2} = (1+a^{-1})\sqrt{1-c^2}-\lambda.$$

This is equivalent to the quadratic equation

$$2\sqrt{1-c^2}a + \left(\lambda + 3\sqrt{1-c^2}\right)a - \sqrt{1-c^2} = 0$$

which has unique positive solution

$$a = \frac{-\left(\lambda+3\sqrt{1-c^2}\right) + \sqrt{\lambda^2 + 6\lambda\sqrt{1-c^2} + 17(1-c^2)}}{4\sqrt{1-c^2}}.$$

We use this to compute $(2a+4)\sqrt{1-c^2}$ and the result follows. □

3.4 Optimal domains and Morse theory

In this section we study more carefully the bilinear operator $D^2\mathcal{F}_\Omega(\bar{g})$ in the case that Ω is a geodesic ball. We recall that

$$\begin{aligned} D^2\mathcal{F}_\Omega(\bar{g})(h,h) = &-\int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}|d\operatorname{tr} h|^2 + |h|^2 + (\operatorname{tr} h)^2\right) \\ &+ \left(\sqrt{1-c^2} - \frac{nc^2}{\sqrt{1-c^2}}\right)\int_\Sigma |X|^2 \\ &+ \left(2\sqrt{1-c^2} - \frac{(n-1)c^2}{2\sqrt{1-c^2}}\right)\int_\Sigma h(N,N)^2, \end{aligned}$$

where $X = (h\cdot N)_\Sigma$. It was shown above that the local scalar curvature rigidity theorem holds on Ω provided that $D^2\mathcal{F}_\Omega(\bar{g})$ is negative definite, or equivalently that the bilinear form $B := -D^2\mathcal{F}_\Omega(\bar{g})$ is positive definite. For each geodesic ball $\Omega = \{f \geq c\}$ we define the spaces

$$\mathcal{X}_c := \left\{h \in W^{1,2} : \|h\|_{L^2} = 1\right\}$$

$$\widehat{\mathcal{X}}_c := \left\{h \in W^{1,2} : \bar{\delta} h = 0, h_\Sigma = 0 \text{ and } \|h\|_{L^2} = 1\right\}$$

and set

$$\mu(c) := \inf\left\{B(h,h) : h \in \mathcal{X}_c\right\},$$

$$\widehat{\mu}(c) := \inf\left\{B(h,h) : h \in \widehat{\mathcal{X}}_c\right\}.$$

Our goal is then to investigate the conditions under which $\widehat{\mu}(c) > 0$. Since $\widehat{\mu}(c) \geq \mu(c)$, we can obtain rigidity results from the condition $\mu(c) > 0$, but they may not be as strong as those derived from $\widehat{\mu}$. However, we mostly focus on μ to simplify the analysis.

For convenience we define

$$\alpha(c) := \frac{nc^2}{\sqrt{1-c^2}} - \sqrt{1-c^2}$$

$$\beta(c) := \frac{(n-1)c^2}{2\sqrt{1-c^2}} - 2\sqrt{1-c^2}$$

so that B can be written

$$B(h,h) = \int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}|d\operatorname{tr} h|^2 + |h|^2 + (\operatorname{tr} h)^2\right) + \alpha(c)\int_\Sigma |X|^2 + \beta(c)\int_\Sigma h(N,N)^2.$$

It is clear that $\widehat{\mu}(c) > 0$ for any value of c having both $\alpha(c) \geq 0$ and $\beta(c) \geq 0$. This occurs precisely when

$$c \geq \frac{2}{\sqrt{n+3}};$$

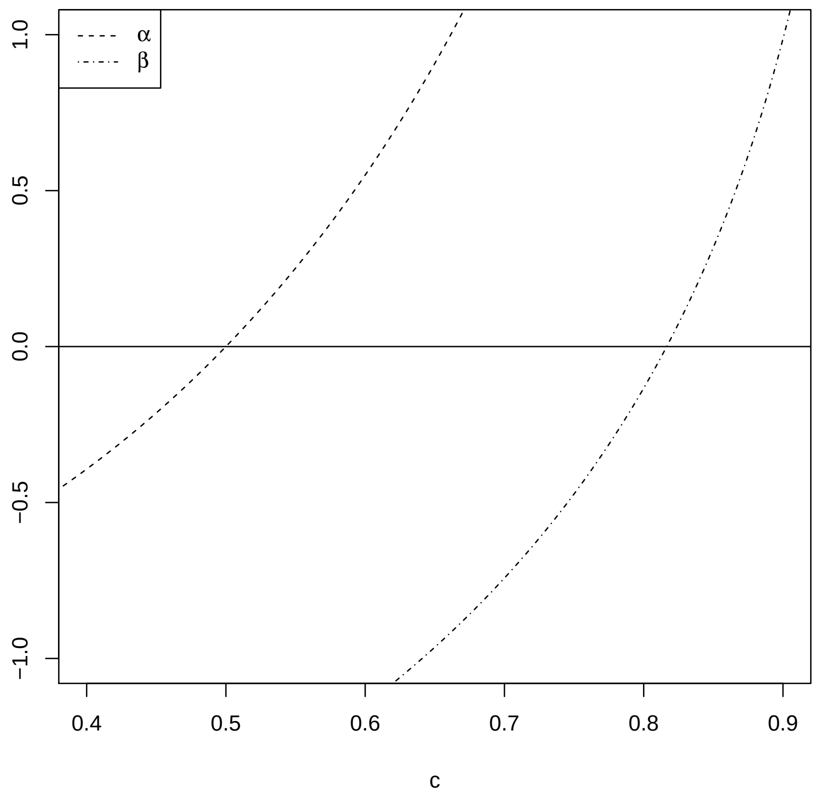

FIGURE 3.1: Boundary coefficients $\alpha(c)$ and $\beta(c)$ plotted as functions of the height c of the domain $\Omega = \{f \geq c\}$, in the case $n = 3$.

this is the result of Brendle and Marques (2010). Figure 3.1 illustrates the dependence of α and β on c when $n = 3$. In particular, we see that α and β are monotone increasing, and are both positive when $c \geq \sqrt{\frac{2}{3}} \approx 0.81$.

Having defined μ and $\widehat{\mu}$, we next reinterpret the main estimate used in the proof of Theorem 22.

Proposition 54. *If c satisfies the bound of Theorem 22, then $\widehat{\mu}(c) > 0$.*

Proof. In the proof of Theorem 22 we found that

$$B(h,h) \geq \frac{1}{2}\int_\Omega \left[f|\nabla h|^2 + 2(f-c)|h|^2\right]$$

for all $h \in \widehat{\mathcal{X}}_c$, so $\widehat{\mu}(c)$ is certainly nonnegative. Now suppose $\widehat{\mu}(c) = 0$, so there exists a sequence $\{h^i\}_{i=1}^{\infty}$ in $\widehat{\mathcal{X}}_c$ with $B(h^i, h^i) \to 0$ as $i \to \infty$. Since

$$\int_\Omega |\nabla h|^2 \leq \frac{1}{c} \int_\Omega f|\nabla h|^2 \leq \frac{2}{c} B(h,h)$$

we see that the sequence $\{h^i\}$ is bounded in $W^{1,2}$. It follows that there is a weakly convergent subsequence

$$h^i \rightharpoonup v \text{ in } W^{1,2}(\Omega),$$

with $h^i \to v$ in $L^2(\Omega)$. Since norms are lower semicontinuous with respect to weak convergence, we conclude that the functional

$$F(h) := \frac{1}{2} \int_\Omega \left[f|\nabla h|^2 + 2(f-c)|h|^2 \right]$$

is lower semicontinuous with respect to weak $W^{1,2}$ convergence. Since $F(h) \leq B(h,h)$, we then have

$$\begin{aligned} F(v) &\leq \liminf F(h^i) \\ &\leq \liminf B(h^i, h^i) \\ &= 0, \end{aligned}$$

hence $v = 0$. However, we also have $h^i \to v$ in L^2, and $\|h^i\|_{L^2} = 1$ for all i. This implies $\|v\|_{L^2} = 1$, a contradiction. □

In particular, we see from the above proposition that $\widehat{\mu}(c) > 0$ when c is sufficiently large. Since the local rigidity theorem fails for the entire hemisphere by the work of Brendle et al. (2010), we must have $\widehat{\mu}(0) \leq 0$.

We now turn our focus to the simpler quantity $\mu(c)$. A straightforward variational computation, together with elliptic regularity, yields the following properties that must be satisfied by any minimizer of B.

Proposition 55. *Let $c \in (0,1)$. Then $h \in \mathcal{X}_c$ satisfies $B(h,h) = \mu(c)$ if and only if it is a smooth solution of the boundary-value problem*

$$\delta\left(f\nabla\left[h + (\operatorname{tr} h)g\right]\right) = 2\mu(c)h \text{ in } \Omega,$$

$$c\nabla_N\left[h + (\operatorname{tr} h)g\right] = \alpha(c)\left[X \otimes N + N \otimes X\right] + 2\beta(c)h(N,N)N \otimes N \text{ on } \Sigma,$$

where $X = (h \cdot N)_\Sigma$.

Proof. Let $v \in \mathcal{X}_c$. By assumption, the function

$$\sigma(t) := \frac{B(h+tv, h+tv)}{\|h+tv\|_{L^2}^2}$$

satisfies $\sigma(0) = \mu(c)$ and $\sigma'(0) = 0$. Differentiating, we find that

$$0 = \int_\Omega f\left(\frac{1}{2}\langle\nabla h, \nabla v\rangle + \frac{1}{2}\langle\nabla \operatorname{tr} h, \nabla \operatorname{tr} v\rangle + \langle h, v\rangle + (\operatorname{tr} h)(\operatorname{tr} v)\right)$$
$$+ \int_\Sigma \left[\alpha(c)\left\langle (h\cdot N)_\Sigma, (v \cdot N)_\Sigma\right\rangle + \beta(c)h(N,N)v(N,N)\right] - \mu(c)\int_\Omega \langle h, v\rangle.$$

We observe the integration by parts formulae

$$\int_\Omega f\langle\nabla h, \nabla v\rangle = -\int_\Omega \left(f\langle\Delta h, v\rangle + \langle\nabla_{\nabla f} h, v\rangle\right) - c\int_\Sigma \langle\nabla_N h, v\rangle,$$

$$\int_\Omega f\langle\nabla \operatorname{tr} h, \nabla \operatorname{tr} v\rangle = -\int_\Omega \left[f(\Delta \operatorname{tr} h)\operatorname{tr} v + (\nabla_{\nabla f} \operatorname{tr} h)\operatorname{tr} v\right] - c\int_\Sigma (\nabla_N \operatorname{tr} h)\operatorname{tr} v,$$

to conclude that, because v was arbitrary,

$$f\Delta\left[h + (\operatorname{tr} h)g\right] + \nabla_{\nabla f}\left[h + (\operatorname{tr} h)g\right] - 2\left[h + (\operatorname{tr} h)g\right] = -2\mu(c)h$$

in Ω. We rewrite this as

$$\delta\left(f\nabla\left[h + (\operatorname{tr} h)g\right]\right) = 2\mu(c)h.$$

We similarly find

$$c\nabla_N\left[h + (\operatorname{tr} h)g\right] = \alpha(c)\left[X \otimes N + N \otimes X\right] + 2\beta(c)h(N,N)N \otimes N$$

on the boundary, hence h is a weak solution to the stated boundary-value problem. Since we have

$$B(h, v) = \mu(c)(h, v)_{L^2}$$

for all $v \in W^{1,2}$, the smoothness of h will follow from elliptic regularity (Theorem 7.32 of Folland (1995)) once the coercivity of B over $W^{1,2}(\Omega)$ has been established. Thus we must show that the estimate

$$B(h, h) \geq C\|h\|_{W^{1,2}} - \lambda\|h\|_{L^2}$$

holds for all $h \in W^{1,2}$, for some constants $C > 0$ and $\lambda \in \mathbb{R}$.

Since $f \geq c > 0$ in Ω, the only concern is the boundary terms occurring in the definition of B. We assume at least one of $\alpha(c)$ and $\beta(c)$ is negative, as the result is trivial otherwise.

Recalling that $\nabla f = \sqrt{1 - c^2}N$ on Σ, we define the one-form

$$\omega := h(\nabla f, \nabla f)h(\nabla f, \cdot)$$

such that $\omega(N) = (1-c^2)^{3/2}h(N, N)^2$ on Σ. We then use the fact that both $|\nabla f|$ and Δf are bounded above and below (away from zero), together with the arithmetic-geometric mean inequality, to find that

$$\begin{aligned} |\delta\omega| &\leq K\left(|h|^2 + |h||\nabla h|\right) \\ &\leq K\left(\frac{1 + 4\epsilon}{4\epsilon}|h|^2 + \epsilon|\nabla h|^2\right) \end{aligned}$$

for some positive constant $K = K(n, c)$, and any $\epsilon > 0$. Now, assuming $\alpha(c) < 0$, the divergence theorem implies

$$\alpha(c)\int_\Sigma h(N, N)^2 \geq K'\int_\Omega\left(\frac{1 + 4\epsilon}{4\epsilon}|h|^2 + \epsilon|\nabla h|^2\right)$$

for some negative constant K' depending only on n and c. If $\alpha(c) \geq 0$ the same estimate holds trivially with $K' = 0$. We similarly find that

$$\beta(c)\int_\Sigma |X|^2 \geq K''\int_\Omega\left(\frac{1 + 4\epsilon}{4\epsilon}|h|^2 + \epsilon|\nabla h|^2\right)$$

for some nonpositive constant K''. Discarding the trace terms in B, we have

$$B(h,h) \geq \int_\Omega \left\{ \left[\frac{c}{2} + \epsilon(K' + K'')\right] |\nabla h|^2 + \left[c + (K' + K'')\left(\frac{1+4\epsilon}{4\epsilon}\right)\right] |h|^2 \right\}$$

for any positive ϵ. We then choose ϵ sufficiently small to ensure

$$\frac{c}{2} + \epsilon(K' + K'') > 0$$

and the desired estimate follows. □

Better understanding of local scalar curvature rigidity phenomena should result from a careful study of the above boundary-value problem—in particular the case $\mu(c) = 0$, as this is the critical point beyond which our methods no longer apply. It is thus important to establish that minimizers in fact exist, as we now verify.

Proposition 56. *For each $c \in (0,1)$ there exists $h \in \mathcal{X}_c$ satisfying $B(h,h) = \mu(c)$.*

Proof. We construct a minimizer by the direct method in the calculus of variations. We thus begin by choosing a sequence $\{h^i\} \in \mathcal{X}_c$ with $B(h^i, h^i) \to \mu(c)$. Without loss of generality we can assume that $B(h^i, h^i) \leq \mu(c) + 1$ for all i. Then the coercivity estimate found in the proof of Proposition 55 implies

$$\begin{aligned} C\|h^i\|_{W^{1,2}} &\leq B(h^i, h^i) + \lambda \|h^i\|_{L^2} \\ &\leq 1 + \lambda + \mu(c) \end{aligned}$$

for all i, so the sequence $\{h^i\}$ is bounded in $W^{1,2}$. It follows that there is a subsequence such that

$$h^i \rightharpoonup h \text{ weakly in } W^{1,2},$$

$$h^i \to h \text{ strongly in } L^2,$$

for some $h \in W^{1,2}$. We next claim that B is weakly lower semicontinuous; it follows from this that

$$\begin{aligned} B(h,h) &\leq \liminf B(h^i, h^i) \\ &= \mu(c). \end{aligned}$$

Because h^i converges to h strongly in L^2, we have $\|h\|_{L^2} = 1$; this means $h \in \mathcal{X}_c$, and so $B(h,h) \geq \mu(c)$. Combining the two previous inequalities, we find that $B(h,h) = \mu(c)$ as desired.

It thus remains to establish the weak lower semicontinuity of B. As in the previous proposition, the only difficulty comes from the boundary terms, due to the unknown sign of the coefficients $\alpha(c)$ and $\beta(c)$. Since norms are lower semicontinuous, we find immediately that the interior terms

$$\int_\Omega f\left(\frac{1}{2}|\nabla h|^2 + \frac{1}{2}|d\,\mathrm{tr}\,h|^2 + |h|^2 + (\mathrm{tr}\,h)^2\right)$$

have the desired continuity property. We will deal explicitly with the $h(N,N)^2$ boundary term; the $|X|^2$ term is dispensed in a similar fashion. As in Proposition 55 we apply the divergence theorem to the one-form

$$\omega := h(\nabla f, \nabla f) h(\nabla f, \cdot)$$

to find

$$(1-c^2)^{3/2}\int_\Sigma h(N,N)^2 = \int_\Omega \delta\omega.$$

The divergence of ω will involves terms of the form $h * h$ and $h * \nabla h$. Specifically, there are no terms quadratic in ∇h. The weak lower semicontinuity of the boundary term is then an immediate consequence of the following lemma, applied to $x_i = \nabla h^i$ and $y_i = h^i$. $\square$

Lemma 57. *Consider sequences $\{x_i\}$ and $\{y_i\}$ in a Hilbert space H such that $x_i \rightharpoonup x$ weakly in H and $y_i \to y$ strongly in H. Then*

$$\langle x_i, y_i\rangle \to \langle x, y\rangle$$

as $i \to \infty$.

Proof. We begin by writing

$$\langle x_i, y_i \rangle - \langle x, y \rangle = \langle x_i, y_i - y \rangle + \langle x_i - x, y \rangle .$$

Now the weak convergence of $\{x_i\}$ implies $\langle x_i - x, y \rangle \to 0$. Since weakly convergent sequences are bounded, we have $\|x_i\| \leq M$ for all i, hence

$$|\langle x_i, y_i - y \rangle| \leq M \|y_i - y\| \to 0$$

by the Cauchy–Schwartz inequality. The result follows. □

4

Rigidity for nonsmooth Euclidean domains

The main goal of this chapter is to prove a version of the Miao–Shi–Tam theorem for compact manifolds with nonsmooth Riemannian metrics. We are interested in metrics on the unit ball that have the appropriate boundary behavior, but are *a priori* only known to be of class C^1, and as such do not have well-defined curvature. We consider limits of smooth metrics with nonnegative curvature, with the following result.

Theorem 58. *Let g_i be a sequence of smooth Riemannian metrics on the unit ball $B \subset \mathbb{R}^n$, with $\mathrm{Ric}(g_i) \geq 0$ and $g_i \to g$ in $C^{1,\alpha}(B)$ for some $0 < \alpha < 1$. If g agrees with the flat Euclidean metric, g_0, in some neighborhood of ∂B, then g is smoothly isometric to g_0. Moreover, if $\mathrm{Ric}(g_i) \leq C$ for all i, then $g_i \to g$ in $W^{2,p}(B)$ for any $p \geq 1$.*

The boundary conditions in the above theorem are likely not optimal. Similar conclusions should be possible with weaker assumptions on the boundary behavior of g, expressed in terms of the mean curvature or second fundamental form. However, the above result suffices for the application of the next chapter, in which we require

$W^{2,p}$ convergence of the sequence for some $p > n$.

Before moving on, we note the following theorem, which also deals with curvature rigidity in the case of a possibly nonsmooth limit.

Theorem 59. *(Bray and Finster (2002)) Suppose $\{g_i\}$ is a sequence of smooth, complete, asymptotically flat metrics on M^3 with nonnegative scalar curvature and total masses $\{m_i\}$ which converge to a (possibly non-smooth) limit metric g in the C^0 sense. Let U be the interior of the set of points where this convergence of metrics is locally C^3 and nondegenerate. Then if the metrics $\{g_i\}$ have uniformly positive isoperimetric constants and their masses $\{m_i\}$ converge to zero, the limit metric g is flat in U.*

By contrast with Theorem 58, this theorem controls the interior geometry in terms of the ADM mass, rather than the boundary geometry of a compact domain. The rigidity statement, however, requires C^3 convergence, and as such is not applicable for the considerably weaker case of $C^{1,\alpha}$ convergence considered here.

4.1 A nonsmooth Bochner-type estimate

In this section we derive a Bochner-type estimate for C^1 metrics. This arises from the classical Bochner formula, and a straightforward limiting argument.

Proposition 60. *Let g be a Riemannian metric of class C^1, on a domain Ω with smooth boundary Σ. Suppose there exists a sequence of smooth metrics g_i, each having $\mathrm{Ric}(g_i) \geq 0$, with $g_i \to g$ in $C^1(\Omega)$. Then for any g-harmonic function $u \in C^2(\Omega)$ we have*

$$\int_\Omega |\nabla du|^2 + \frac{1}{2}\int_\Sigma \nabla_N |du|^2 \leq 0,$$

where N denotes the inward unit normal to Σ.

Proof. We let $\eta = du$ for convenience. Then for each g_i, the Bochner formula for one-forms says that

$$\Delta_H \eta = \nabla^* \nabla \eta + \mathrm{Ric}(\eta, \cdot),$$

where $\Delta_H = dd^* + d^*d$ denotes the Hodge Laplacian. We then take the inner product with η and integrate, to find

$$\int_\Omega \langle \Delta_H \eta, \eta \rangle \geq \int_\Omega \langle \nabla^* \nabla \eta, \eta \rangle \,, \tag{4.1}$$

for each g_i, using the fact that $\mathrm{Ric} \geq 0$. Integration by parts yields

$$\int_\Omega \langle \nabla^* \nabla \eta, \eta \rangle = \int_\Omega |\nabla \eta|^2 + \int_\Sigma \langle \nabla_N \eta, \eta \rangle$$

and

$$\begin{aligned} \int_\Omega \langle dd^* \eta, \eta \rangle &= \int_\Omega |d^* \eta|^2 - \int_\Sigma (d^* \eta) \langle \eta, N \rangle \\ &= \int_\Omega (\Delta u)^2 + \int_\Sigma (\Delta u) \nabla_N u \end{aligned}$$

where we have recalled that $d^* \eta = d^* du = -\Delta u$. Since $d\eta = d^2 u = 0$, we find from Equation (4.1) that

$$\int_\Omega (\Delta u)^2 + \int_\Sigma (\Delta u) \nabla_N u \geq \int_\Omega |\nabla du|^2 + \int_\Sigma \langle \nabla_N du, du \rangle$$

for all i. The terms in the above inequality depend continuously on the metric and its first derivatives, so we can take the limit $i \to \infty$ to conclude that this inequality also holds for the limiting metric g. Since $\Delta^g u = 0$ by assumption, the result follows. □

The following corollary gives the inequality needed to prove Theorem 58. In this case g agrees with g_0 in some neighborhood of ∂B, so it can be extended to a $C^{1,\alpha}$ metric $\tilde{g}$ on $\mathbb{R}^n$.

Corollary 61. *Consider $\{g_i\}$ and g as in Theorem 58, and let $\tilde{g}$ denote the extension of g to $\mathbb{R}^n$. Then for any $\tilde{g}$-harmonic function u we have*

$$\int_\Omega |\nabla du|^2 + \frac{1}{2}\int_{\partial\Omega} \nabla_N |du|^2 \leq 0$$

on any domain $\Omega \subset \mathbb{R}^n$ containing B.

Proof. We let N denote the inward unit normal to $\partial\Omega$ and ν the inward unit normal to ∂B. Then Proposition 60 implies

$$\int_B |\nabla du|^2 + \frac{1}{2}\int_{\partial B} \nabla_\nu |du|^2 \leq 0.$$

Since $\tilde{g}$ is flat outside B, we can apply the classical Bochner formula and integrate over $\Omega \setminus B$ to obtain

$$0 = \int_{\Omega\setminus B} |\nabla du|^2 + \frac{1}{2}\int_{\partial\Omega} \nabla_N |du|^2 - \frac{1}{2}\int_{\partial B} \nabla_\nu |du|^2 .$$

We then sum these equations and the result follows. □

4.2 Proof of rigidity

In this section we prove the first half of Theorem 58. The proof uses some ideas from the theory of weighted Hölder spaces on asymptotically flat manifolds, as reviewed in Appendix B.

Since g agrees with g_0 near the boundary, we can trivially extend it to a $C^{1,\alpha}$ metric on $\mathbb{R}^n$ that is flat outside of a bounded set. Thus the extended metric, $\tilde{g}$, is asymptotically flat and has

$$m_{\mathrm{ADM}}(\tilde{g}) = 0.$$

We let $x^1, \ldots, x^n$ denote the standard Euclidean coordinates on $\mathbb{R}^n$, and fix a constant $\delta > 0$. It follows from Theorem 83 that there exist functions $u^1, \ldots, u^n$ of class $C^{2,\alpha}$

satisfying

$$\Delta u^a = 0 \text{ in } \mathbb{R}^n,$$

$$u^a - x^a \in C^{2,\alpha}_{\delta}(\mathbb{R}^n),$$

for $1 \le a \le n$. Elliptic regularity implies that the functions u^a are in fact smooth outside of a bounded set.

We next apply Corollary 60 to each u^a and sum to find

$$\sum_{a=1}^n \int_{B_R} |\nabla du^a|^2 + \frac{1}{2} \sum_{a=1}^n \int_{S_R} \nabla_N |du^a|^2 \le 0,$$

where $B_R \subset \mathbb{R}^n$ is a geodesic ball of radius R, with boundary S_R. We then have from Lemma 84 that

$$\lim_{R\to\infty} \left(\sum_{a=1}^n \int_{S_R} \nabla_N |du^a|^2 \right) = -c_n^{-1} m_{\mathrm{ADM}}(\tilde{g})$$

for some positive constant c_n, and so

$$\int_{\mathbb{R}^n} |\nabla du^a|^2 = 0$$

for $1 \le a \le n$. Thus the one-forms $du^1, \ldots, du^n$ are parallel. Now the condition $u^a - x^a \in C^{2,\alpha}_{-\delta}(\mathbb{R}^n)$ implies

$$\lim_{|x|\to\infty} \left\langle du^a, du^b \right\rangle (x) = \delta^{ab},$$

so we in fact have that $du^1, \ldots, du^n$ are globally orthonormal.

We have thus far produced a set of globally defined coordinates in which $g_{ij} = \delta_{ij}$. However, these coordinates, and hence the resulting diffeomorphism between $\tilde{g}$ and g_0, are only known to be of class $C^{2,\alpha}$. Thus it remains to be shown that $\tilde{g}$ is smooth. This is an immediate consequence of the following regularity theorem.

Proposition 62. *Let M be a smooth manifold with a $C^{1,\alpha}$ Riemannian metric g. Suppose there exist harmonic coordinates $\{u^a\}$ on an open subset $U \subset M$ such that* *the Christoffel symbols* Γ^c_{ab} *are of class* $C^{k,\alpha}$ *for some* $k \geq 1$*. Then the coordinates* $\{u^a\}$ *are of class* $C^{k+2,\alpha}$*, and* $g|_U$ *is of class* $C^{k+1,\alpha}$*.*

We will make use of the following regularity result of Kazdan and DeTurck, which is a consequence of the fact that the map $g \mapsto \Gamma$ (taking a metric g to the corresponding Christoffel symbols) is an overdetermined elliptic operator.

Lemma 63. *(DeTurck and Kazdan (1981)) Let* g *be a* C^1 *metric. If in some local coordinates* Γ^c_{ab} *is of class* $C^{k,\alpha}$ *for some* $k \geq 0$*, then in these coordinates the metric* g *is of class* $C^{k+1,\alpha}$*.*

This differs from the proposition above in that it only asserts regularity of g in the given coordinates. The ideas behind Proposition 62 is to use the harmonicity assumption to deduce improved regularity of the $\{u^a\}$ coordinates, and hence conclude that g is of class $C^{k+1,\alpha}$ with respect to *any* smooth coordinate chart.

Proof. (of Proposition 62) Since g is $C^{1,\alpha}$, the coefficients of Δ will be of class C^α in any coordinate chart. Standard regularity theory then implies that the harmonic coordinates $\{u^a\}$ are $C^{2,\alpha}$.

We now proceed by induction, starting with the case $k = 1$. We first recall the coordinate expression

$$(\mathrm{Hess}\, f)_{ab} = \frac{\partial^2 f}{\partial u^a \partial u^b} - \Gamma^c_{ab} \frac{\partial f}{\partial u^c}$$

for any C^2 function f. Now in any (smooth) coordinate chart $\{x^i\}$ we have

$$(\mathrm{Hess}\, f)_{ij} = \frac{\partial u^a}{\partial x^i} \frac{\partial u^b}{\partial x^j} (\mathrm{Hess}\, f)_{ab} \tag{4.2}$$

so we see that Δ has $C^{1,\alpha}$ coefficients. Then elliptic regularity implies that the coordinate functions $\{u^a\}$ are in fact $C^{3,\alpha}$. We have from Lemma 63 that $g_{ab} \in C^{2,\alpha}$, hence the equation

$$g_{ij} = \frac{\partial u^a}{\partial x^i} \frac{\partial u^b}{\partial x^j} g_{ab} \tag{4.3}$$

implies that g is of class $C^{2,\alpha}$.

For the inductive step we let $k > 1$ and assume that the result holds for all $1 \le j \le k-1$. We are given that Γ^c_{ab} is $C^{k,\alpha}$, so the result for $k-1$ implies that g is $C^{k,\alpha}$, whence the coordinates $\{u^a\}$ are $C^{k+1,\alpha}$. It follows from (4.2) that the coefficients of Δ are $C^{k,\alpha}$, hence the $\{u^a\}$ are in fact $C^{k+2,\alpha}$. Then Lemma 63 implies that g_{ab} is $C^{k+1,\alpha}$, and it follows from (4.3) that g is of class $C^{k+1,\alpha}$. □

In the proof of Theorem 58 we had $g_{ab} = \delta_{ab}$, hence $\Gamma^c_{ab} = 0$. Thus Proposition 62 immediately implies the smoothness of g with respect to the usual differentiable structure of $\mathbb{R}^n$.

4.3 Improved Sobolev convergence

We next turn to the second half of Theorem 58. We recall that $g_i \to g$ in $C^{1,\alpha}(B)$, and g is smoothly isometric to the Euclidean metric g_0. Additionally, we are assuming the uniform bounds

$$0 \le \mathrm{Ric}(g_i) \le C. \tag{4.4}$$

Our goal is to prove that $g_i \to g$ in $W^{2,p}(B)$ when $p \ge 1$.

To accomplish this we will use harmonic coordinates on B. While it it known that any Riemannian metric admits harmonic coordinates locally, we need to ensure that this can be done uniformly in i; in other words we want each $x \in B$ to be contained in a neighborhood U_x such that there exist g_i-harmonic coordinates on U_x for every i. The results in Appendix A show that this is possible given uniform bounds on the Ricci curvature and injectivity radius. The Ricci curvature bound is precisely (4.4), and the uniform bound on injectivity radii follows from the $C^{1,\alpha}$ convergence of the g_i to the Euclidean metric.

Lemma 64. *Let Ω be a compact manifold with boundary Σ. Suppose $\{g_i\}$ and g are* *smooth Riemannian metrics, with $g_i \to g$ in C^1. Then*

$$\lim_{i\to\infty} \int_B R(g_i) = \int_B R(g).$$

Proof. We first partition the domain as

$$\Omega = U_1 \cup \cdots \cup U_N,$$

where each U_j is diffeomorphic to a smooth domain with corners in $\mathbb{R}^n$, and the U_j only intersect along their mutual boundaries. Then

$$\int_\Omega f = \sum_{j=1}^{N} \int_{U_j} f$$

for any integrable function f on Ω, so it suffices to prove the lemma on each U_j. We thus let $U = U_j$ for some j.

Since local coordinates exist on U we can write the scalar curvature as

$$R = \delta Z + s,$$

where Z and s are smooth functions of the metric and its first derivatives (see Bartnik (1986) for details). Integrating by parts, we have

$$\int_U R(g_i) = \int_U s_i + \int_{\partial U} \langle Z_i, N_i \rangle ,$$

where N_i denotes the inward unit normal to ∂U computed with respect to g_i. Since the right-hand side of the above equation depends continuously on g and its first derivatives, we have

$$\begin{aligned} \lim_{i\to\infty} \left(\int_U s_i + \int_{\partial U} \langle Z_i, N_i \rangle \right) &= \int_U s + \int_{\partial U} \langle Z, N \rangle \\ &= \int_U R(g) \end{aligned}$$

as desired. □

We therefore have $\int_\Omega R(g_i) \to 0$ as $i \to \infty$. We next use the assumption of nonnegative Ricci curvature to get L^p convergence of the Ricci tensor.

Lemma 65. *Assume the hypotheses of Theorem 58. Then*

$$\lim_{i\to\infty} \| \operatorname{Ric}(g_i)\|_{L^p(\Omega)} = 0.$$

for all $p \geq 1$.

Proof. The inequality $\operatorname{Ric}(g_i) \geq 0$ implies $|\operatorname{Ric}(g_i)| \leq R(g_i)$. Taking the p^{th} power and recalling that $|\operatorname{Ric}(g_i)| \leq \sqrt{n}C$, we obtain

$$0 \leq |\operatorname{Ric}(g_i)|^p \leq \left(\sqrt{n}C\right)^{p-1} R(g_i).$$

Now the fact that $R(g_i) \to 0$ in L^1 immediately implies

$$\lim_{i\to\infty} \int_\Omega |\operatorname{Ric}(g_i)|^p = 0$$

as claimed. □

Using these lemmata, together with appropriately chosen harmonic coordinates, we complete the proof of $W^{2,p}$ convergence.

Proposition 66. *Assume the hypotheses of Theorem 58. Then* $g_i \to g$ *in* $W^{2,p}$ *for all* $p \geq 1$.

Proof. For this proof will use i exclusively to index sequences, and use a and b as coordinates indices. We first prove the result assuming $p > 1$; since B is bounded the case $p = 1$ immediately follows.

Let $x \in B$. Then for some neighborhood U of x and every i, there exist functions $(u_i^1, \ldots, u_i^n)$ such that:

- $(u_i^1, \ldots, u_i^n)$ are g_i-harmonic coordinates in U;

- $u_i^a \to u^a$ in $C^{2,\alpha}(U)$ for $1 \leq 1 \leq n$, where $(u^1, \dots, u^n)$ are g-harmonic coordinates on U.

Thus there exist harmonic coordinates with respect to each g_i defined on some uniform neighborhood of x. With respect to these coordinates we have

$$-2\,\mathrm{Ric}(g_i)_{ab} = \Delta_{g_i}(g_i)_{ab} + Q_{ab}(g_i^{-1}, \partial g_i)$$

where Q is quadratic in g and ∂g. Since $g_i \to g$ in $C^{1,\alpha}$, we find that

$$Q_{ab}(g_i^{-1}, \partial g_i) \to Q_{ab}(g^{-1}, \partial g)$$

in $C^{1,\alpha}$ and hence in L^p for any $p > 0$. Moreover, it follows from Lemma 65 that $\mathrm{Ric}(g_i)_{ab} \to 0$ in L^p as long as $p \geq 1$, so we can conclude that

$$\Delta_{g_i}(g_i)_{ab} \to \Delta_g g_{ab}$$

in L^p; here we recall that g in fact smooth, so the right-hand side of the above limit is defined. We now apply the standard L^p estimates for $p > 1$ (Theorem 9.11 of Gilbarg and Trudinger (1983)) to find on any compact subset $K \Subset \Omega$

$$\|g_{ab} - (g_i)_{ab}\|_{W^{2,p}(K)} \leq C\left(\|\Delta_{g_i}(g_{ab} - (g_i)_{ab})\|_{L^p(K)} + \|g_{ab} - (g_i)_{ab}\|_{L^p(K)}\right)$$

for some constant C independent of i. We first observe that $g_{ab} - (g_i)_{ab} \to 0$ in L^p. For the remaining term on the right-hand side we have

$$\|\Delta_{g_i}(g_{ab} - (g_i)_{ab})\|_{L^p(K)} \leq \|(\Delta_{g_i} - \Delta_g)g_{ab}\|_{L^p(K)} + \|\Delta_g g_{ab} - \Delta_{g_i}(g_i)_{ab}\|_{L^p(K)}.$$

It was shown above that $\Delta_{g_i}(g_i)_{ab} \to \Delta_g g_{ab}$ in L^p; since $g_i \to g$ in $C^{1,\alpha}$ we also have that $\Delta_{g_i} g_{ab} \to \Delta_g g_{ab}$ in L^p. Combining these results, we conclude that $(g_i)_{ab} \to g_{ab}$ in $W^{2,p}(K)$.

To complete the proof, we observe that there exist neighborhoods $U_1, \dots, U_N$, each of which admits g_i-harmonic coordinates for every i, and compact subsets $K_i \Subset U_i$ with $K_1 \cup \cdots \cup K_N = B$. $\square$

5

Towards a global rigidity theorem

In this final chapter we finally discuss the subject of global scalar curvature rigidity, thus considering metrics that are not necessarily close to the spherical metric in the $W^{2,p}$ topology. We first observe that no global rigidity property holds on the entire hemisphere, by the counterexample of Brendle et al. (2010). It then seems natural to study the problem of global rigidity on domains smaller than the entire hemisphere, as this approach proved fruitful in the local case.

We are currently unable to prove a theorem of this sort, but make a conjecture inspired by the results of Chapter 4. To state the conjecture, we first define $\mathcal{M}(r, \lambda, D, \iota)$ to be the space of metrics on $\mathbb{S}^n$ satisfying

$$\begin{aligned} \operatorname{supp}(g - \bar{g}) &\subset B_r(x) \text{ for some } x, \\ r^2 |\operatorname{Ric}(g)| &\leq \lambda \text{ in } B_r(x), \\ \operatorname{diam}(B_r(x), g) &\leq rD, \\ \operatorname{inj}(g) &\geq r\iota \text{ in } B_r(x), \end{aligned}$$

for positive constants r, λ, D and ι. The convention here is that $B_r(x)$ is always computed with respect to $\bar{g}$, hence $\operatorname{diam}(B_r(x), g)$ is not necessarily equal to $2r$ as suggested by the notation.

The bounds on $\mathrm{Ric}(g)$ and $\mathrm{inj}(g)$ in this definition become weaker as r decreases, so that a sequence $g_i \in \mathcal{M}(r_i, \lambda, \iota)$, with $r_i \to 0$, could in fact have $\mathrm{Ric}(g_i) \to \infty$ or $\mathrm{inj}(g_i) \to 0$. However, when such a sequence is rescaled by r_i (as will be done in Section 5.2) the resulting metrics will have uniformly bounded Ricci curvature, and injectivity radius bounded away from zero.

Conjecture 67. *Let* $\lambda, \epsilon, D, \iota > 0$. *There exists a positive constant* $r = r(n, \lambda, \epsilon, D, \iota)$ *such that any* $g \in \mathcal{M}(r, \lambda, D, \iota)$ *with* $\mathrm{Ric}(g) \geq 0$ *and* $R(g) \geq n(n-1)$ *is diffeomorphic to* $\bar{g}$.

This conjecture is only approximate—for reasons discussed below it might be necessary to assume a slightly stronger bound on the Ricci curvature, such as

$$r^{2-\epsilon}|\,\mathrm{Ric}(g)| \leq \lambda$$

for some positive constant ϵ.

5.1 Evidence for Conjecture 67

In the remainder of this Chapter we discuss a possible method of proof for the conjecture, and explain the technical hurdles to be overcome in completing this program.

We will proceed by contradiction, and thus assume that there exist positive constants λ, D and ι, a sequence of radii $r_i \to 0$, and metrics $g_i \in \mathcal{M}(r_i, \lambda, D, \iota)$ satisfying the hypotheses of the theorem, such that no g_i is diffeomorphic to $\bar{g}$. Without loss of generality we may assume $\mathrm{supp}(g_i - \bar{g}) \subset B_{r_i}(N)$ for each i, where $N \in \mathbb{S}^n$ denotes the north pole. We then carry out the following steps.

1. Rescale $\{g_i\}$ to obtain a sequence of metrics $\{\widetilde{g}_i\}$ on the unit ball $B \subset \mathbb{R}^n$ with uniformly bounded Ricci curvature and injectivity radius.

2. Employ the convergence theory detailed in Appendix A to extract a convergent subsequence $\widetilde{g}_i \to \widetilde{g}$ in $C^{1,\alpha}(B)$ for some $\alpha \in (0,1)$.

3. Prove that $\widetilde{g}$ is flat near ∂B, then apply Theorem 58 to conclude that $\widetilde{g}$ is flat, and $\widetilde{g}_i \to \widetilde{g}$ in $W^{2,p}(B)$ for any $p \geq 1$.

4. Conclude that the original sequence $\{g_i\}$ converges to $\bar{g}$ in $W^{2,p}$.

5. Use Theorem 22 to find that g_i is diffeomorphic to $\bar{g}$ for i sufficiently large, thus achieving the desired contradiction.

As we will see below, the only unresolved issue is the $W^{2,p}$ convergence in Step 4. We can prove that $\widetilde{g}_i \to \widetilde{g}$ in $W^{2,p}(B)$ for any $p \geq 1$ (as in Step 3), but from this are only able to conclude that the original sequence $\{g_i\}$ converges to $\bar{g}$ in $W^{2,p}$ for $1 \leq p \leq \frac{n}{2}$. Theorem 22 is for metrics in a $W^{2,p}$-neighborhood of $\bar{g}$ for any $p > n$, and as such is not applicable here. To resolve this would require some quantitative control on the rate at which $\|\widetilde{g}_i - \widetilde{g}\|_{W^{2,p}(B)}$ converges to zero. For instance, it would suffice to show that

$$\|\widetilde{g}_i - \widetilde{g}\|_{W^{2,p}(B)} \leq C(r_i)^k$$

for $k > 2 - \frac{n}{p}$.

We devote the remainder of the chapter to filling in the details of the above steps.

5.2 Rescaling the sequence of metrics

Suppose g is a metric on $\mathbb{S}^n$, with $\operatorname{supp}(g - \bar{g}) \subset B_\epsilon(N)$ for some $\epsilon < \pi$. Letting r denote the distance from the north pole N, we can write the standard spherical metric as

$$\bar{g} = dr^2 + \sin^2 r d\sigma^2,$$

where $d\sigma^2$ denotes the standard metric on $\mathbb{S}^{n-1}$. Writing $p \in \mathbb{S}^n$ in polar coordinates as (r, θ), we define a diffeomorphism by

$$\begin{aligned} \Psi_\epsilon : B_\epsilon(N) &\longrightarrow B \subset \mathbb{R}^n \\ (r, \theta) &\longmapsto (r/\epsilon, \theta). \end{aligned}$$

Using standard polar coordinates in $\mathbb{R}^n$ we easily see that

$$(\Psi_\epsilon^{-1})^* \bar{g} = \epsilon^2 dr^2 + \sin^2(\epsilon r) d\sigma^2.$$

Now given any metric g on $B_\epsilon(N) \subset \mathbb{S}^n$, we define its ϵ-rescaling to be

$$g^\epsilon := \epsilon^{-2} (\Psi_\epsilon^{-1})^* g$$

on $B \subset \mathbb{R}^n$. It follows immediately that the spherical metric has ϵ-rescaling

$$\bar{g}^\epsilon = dr^2 + \left(\frac{\sin \epsilon r}{\epsilon} \right)^2 d\sigma^2,$$

which converges uniformly—along with all of its derivatives—to the flat metric g_0 on B as $\epsilon \to 0$. Expanding the sine term for small values of ϵ, we find that $\bar{g}^\epsilon - g_0 = \mathcal{O}(\epsilon^2)$, and similarly for all derivatives.

Returning to the proof of Theorem 67, and recalling that $\operatorname{supp}(g_i - \bar{g}) \subset B_r$, we define the rescaled sequence of metrics by $\widetilde{g}_i = (g_i)^{2r_i}$ (assuming, without loss of generality, that $r < \frac{\pi}{2}$). The properties of this rescaling are given by the following proposition.

Proposition 68. *Define $\widetilde{g}_i$ as above. Then the following are true for all i:*

1. *$\widetilde{g}_i \to g_0$ smoothly in the annulus $A := B_1(0) \setminus B_{1/2}(0)$;*

2. *$|\operatorname{Ric}(\widetilde{g}_i)| \leq \max\{4\lambda, \sqrt{n}(n-1)\pi^2\}$;*

3. *$\operatorname{diam}(B, \widetilde{g}_i) \leq 1 + \frac{D}{2}$;*

4. *$\operatorname{inj}(\widetilde{g}_i) \geq \iota$.*

Proof. Since g_i was rescaled by a factor of $2r$ rather than r, we have $(g_i)^{2r} = \bar{g}^{2r}$ on A, and the first result follows.

To prove the Ricci curvature bound, we first observe from the diffeomorphism invariance and scale invariance of the Ricci curvature tensor that

$$\begin{aligned} |\mathrm{Ric}(\widetilde{g}_i)|^2_{\widetilde{g}_i} &= (2r)^4 \, |\mathrm{Ric}(g_i)|^2_{g_i} \\ &\leq 16\lambda^2, \end{aligned}$$

and so $|\mathrm{Ric}(\widetilde{g}_i)|^2 \leq 4\lambda$ in $B_{1/2}$. In the annular region we can explicitly compute $|\mathrm{Ric}(\widetilde{g}_i)|^2 = (2r)^4 n(n-1)^2$. We then use the fact that $r < \frac{\pi}{2}$ and the result follows.

For the diameter bound, we similarly have

$$\mathrm{diam}(B, \widetilde{g}_i) = \frac{1}{2r} \, \mathrm{diam}\left(B_{2r}(N), g_i\right).$$

We then apply the triangle inequality to find that $\mathrm{diam}(B_{2r}(N), g_i) \leq 2r + rD$ and the result follows.

The proof of the injectivity radius bound is similar. □

5.3 Finding a convergent subsequence

In the previous section we constructed a sequence $\{\widetilde{g}_i\}$ of smooth metrics on the unit ball that had uniformly bounded Ricci curvature, diameter and injectivity radius. We next prove the existence of a convergent subsequence having the desired boundary behavior.

Proposition 69. *Let $\alpha \in (0,1)$. Then there exists a metric $\tilde{g}$ on B such that*

$$\widetilde{g}_i \to \widetilde{g}$$

subsequentially in $C^{1,\alpha}(B)$. Moreover, $\widetilde{g}$ agrees with g_0 in some neighborhood of ∂B.

Proof. It follows immediately from Anderson's convergence theorem that there exists a metric g' on B, and diffeomorphisms $\varphi_i : B \to B$, such that

$$\varphi_i^* \widetilde{g}_i \to g'$$

subsequentially in $C^{1,\alpha}(B)$. It is a consequence of Anderson's proof (*cf.* Theorem 2.2 in Petersen (1997)) that the sequence $\{\varphi_i\}$ converges in $C^{2,\alpha}(B)$ to some limiting diffeomorphism φ on B. Let $\psi = \varphi^{-1}$ and define $\widetilde{g} = \psi^* g'$, so that

$$(\varphi_i \circ \psi)^* \widetilde{g}_i \to \widetilde{g}.$$

We then write

$$\widetilde{g}_i - \widetilde{g} = (id - \varphi_i \circ \psi)^* \, \widetilde{g}_i + [(\varphi_i \circ \psi)^* \widetilde{g}_i - \widetilde{g}] \, .$$

Since $\varphi_i \circ \psi$ converges to the identity in $C^{2,\alpha}(B)$, we conclude that $\widetilde{g}_i \to \widetilde{g}$ in $C^{1,\alpha}(B)$.

It remains to be shown that $\widetilde{g}$ agrees with g_0 near the boundary. This is trivial, because $\widetilde{g}_i$ was chosen to converge smoothly to g_0 in the annular region $A = B_1 \backslash B_{1/2}$. This completes the proof. □

Since the rescaled metrics $\widetilde{g}_i$ all had nonnegative Ricci curvature, we can now appeal to Theorem 58 to conclude that $\widetilde{g}$ is in fact isometric to g_0; we thus write $g_0 = \tau^* \widetilde{g}$ for some diffeomorphism τ that restricts to the identity on the annular region A. Moreover, the curvature and injectivity radius bounds from Proposition 68 allow us to conclude that

$$\tau^* \widetilde{g}_i \longrightarrow g_0 \tag{5.1}$$

in $W^{2,p}(B)$ for any $p \geq 1$.

5.4 Back to the sphere

We now return to the original sequence $\{g_i\}$ of metrics on $\mathbb{S}^n$. The idea is to use the fact that the rescaled sequence converges to g_0 in B (up to a diffeomorphism) to control the $W^{2,p}$ norm of $g_i - \bar{g}$ on $\mathbb{S}^n$.

Proposition 70. *There exists a positive constant K, and a diffeomorphism η_i of $B_{2r_i} \subset \mathbb{S}^n$ for each i, such that*

$$\|\eta_i^* g_i - \bar{g}\|_{W^{2,p}(\mathbb{S}^n, \bar{g})} \leq K (2r_i)^{n/p-2} \left\{ \|\tau^* \widetilde{g}_i - g_0\|_{W^{2,p}(B, g_0)} + r_i^2 \right\} .$$

for sufficiently large i.

For definiteness, we will define the $W^{2,p}$ Sobolev norm on $\mathbb{S}^n$ as

$$\|h\|_{W^{2,p}(\mathbb{S}^n,\bar{g})} = \left[\int_{\mathbb{S}^n} \left(|h|_{\bar{g}}^p + |\overline{\nabla} h|_{\bar{g}}^p + |\overline{\nabla}^2 h|_{\bar{g}}^p\right) d\bar{V}\right]^{1/p}.$$

We similarly define $W^{2,p}(B, g_0)$ and $W^{2,p}(B, (\bar{g})^{2r})$ for the unit ball in $\mathbb{R}^n$. We then have the following lemma relating the $\bar{g}$ and g_0 Sobolev norms on $\mathbb{S}^n$ and B, respectively.

Lemma 71. *There exists a constant K so that*

$$\|h\|_{W^{2,p}(B,\bar{g})} \leq K(2r)^{n/p-4} \left\| \left(\psi_{2r}^{-1}\right)^* h \right\|_{W^{2,p}(B,g_0)}$$

for any symmetric $(0,2)$-tensor h on $B_{2r} \subset \mathbb{S}^n$, provided $r < \frac{\pi}{2}$.

Proof. For h on B_{2r} we have by diffeomorphism invariance that

$$\begin{aligned} \|h\|_{W^{2,p}(B_{2r},\bar{g})} &= \left\| \left(\psi_{2r}^{-1}\right)^* h \right\|_{W^{2,p}\left(B,\left(\psi_{2r}^{-1}\right)^*\bar{g}\right)} \\ &= \left\| \left(\psi_{2r}^{-1}\right)^* h \right\|_{W^{2,p}(B,(2r)^2(\bar{g})^{2r})}, \end{aligned} \tag{5.2}$$

where in the last line we have recalled the definition $g^{2r} := (2r)^{-2} \left(\psi_{2r}^{-1}\right)^* g$. Under a constant rescaling $g \mapsto \lambda^2 g$ we have

$$\begin{aligned} |h|_g^p &\mapsto \lambda^{-2p} |h|_g^p \\ |\nabla h|_g^p &\mapsto \lambda^{-3p} |\nabla h|_g^p \\ |\nabla^2 h|_g^p &\mapsto \lambda^{-4p} |\nabla^2 h|_g^p \\ dV_g &\mapsto \lambda^n dV_g. \end{aligned}$$

Applying these with $\lambda = 2r$, we find that

$$\left\| \left(\psi_{2r}^{-1}\right)^* h \right\|_{W^{2,p}(B,(2r)^2(\bar{g})^{2r})} \leq (2r)^{n/p-4} \left\| \left(\psi_{2r}^{-1}\right)^* h \right\|_{W^{2,p}(B,(\bar{g})^{2r})}, \tag{5.3}$$

where we have used the fact that $\max\{\lambda^{-2p}, \lambda^{-3p}, \lambda^{-4p}\} = \lambda^{-4p}$ as long as $\lambda \leq 1$. Since the rescaled metrics $(\bar{g})^{2r}$ converges smoothly to g_0 as $r \to 0$, there exists a constant C such that

$$\left\| \left(\psi_{2r}^{-1}\right)^* h \right\|_{W^{2,p}(B,(\bar{g})^{2r})} \leq C \left\| \left(\psi_{2r}^{-1}\right)^* h \right\|_{W^{2,p}(B,g_0)} \tag{5.4}$$

for all h and any $r < \frac{\pi}{2}$. We combine (5.2), (5.3) and (5.4) and the result follows. $\square$

With this understanding of the scaling properties of the $W^{2,p}$ norm, we can now prove the main result of the section.

Proof. (of Proposition 70) We define η_i on B_{2r_i} by the composition

$$B_{2r_i} \xrightarrow{\psi_{2r_i}} B \xrightarrow{\tau} B \xrightarrow{\psi_{2r_i}^{-1}} B_{2r_i},$$

where τ is defined in (5.1). Since τ agrees with the identity in the annular region $B \setminus B_{1/2} \subset \mathbb{R}^n$, each η_i will have the same property on $B_{2r_i} \setminus B_{r_i} \subset \mathbb{S}^n$, and hence can be trivially extended to the entire sphere. Now the definition of η_i implies

$$\begin{aligned}
\left(\psi_{2r_i}^{-1}\right)^* (\eta_i^* g_i - \bar{g}) &= \left(\eta_i \circ \psi_{2r_i}^{-1}\right)^* g_i - \left(\psi_{2r_i}^{-1}\right)^* \bar{g} \\
&= \left(\psi_{2r_i}^{-1} \circ \tau\right)^* g_i - (2r_i)^2 (\bar{g})^{2r} \\
&= (2r_i)^2 \left[\tau^* \widetilde{g}_i - (\bar{g})^{2r_i}\right],
\end{aligned}$$

so we can apply Lemma 71 to obtain

$$\|\eta_i^* g_i - \bar{g}\|_{W^{2,p}(B,\bar{g})} \leq K (2r_i)^{n/p-2} \left\|\tau^* \widetilde{g}_i - (\bar{g})^{2r_i}\right\|_{W^{2,p}(B,g_0)} \tag{5.5}$$

for some constant K that does not depend on i or p. From Section 5.2 we have

$$\left\|(\bar{g})^{2r_i} - g_0\right\|_{W^{2,p}(B,g_0)} = \mathcal{O}(r_i^2) \tag{5.6}$$

as $i \to \infty$, so the result follows from (5.5), (5.6) and the triangle inequality. $\square$

5.5 Conclusion

We observe that the second term on the right-hand side of the inequality in Proposition 70 is equal to $K(2r_i)^{n/p}$, and hence converges to zero as $i \to \infty$, for any $p > 0$. The first term is given by

$$(2r_i)^{n/p-2} \|\tau^* \widetilde{g}_i - g_0\|_{W^{2,p}(B,g_0)}.$$

We have already shown that $\tau^* \widetilde{g}_i - g_0 \to 0$ in $W^{2,p}(B, g_0)$ for any $p \geq 1$. Since $(2r_i)^{n/p-2}$ remains bounded precisely when $2p \leq n$, we immediately find the following.

Proposition 72. *Consider $\{g_i\}$ on $\mathbb{S}^n$ as constructed above. Then*

$$\|\eta_i^* g_i - \bar{g}\|_{W^{2,p}(\mathbb{S}^n,\bar{g})} \to 0$$

for any $1 \leq p \leq \frac{n}{2}$.

However, to get convergence for some $p > n$, we will need stronger control over the rate at which

$$\|\tau^* \widetilde{g}_i - g_0\|_{W^{2,p}(B,g_0)}$$

approaches zero.

Appendix A

Convergence of Riemannian metrics with curvature bounds

In this appendix we review the existence and basic properties of harmonic coordinates on a Riemannian manifold. These coordinates, first used in DeTurck and Kazdan (1981) are useful in studying the regularity of nonsmooth metrics. They also make an appearance in the convergence theory of Riemannian manifolds with bounded Ricci curvature (Anderson (1990)).

A.1 Harmonic coordinates

Let (M^n, g) be a Riemannian manifold, with g of class $C^{k,\alpha}$ for some $k \geq 1$ and $\alpha \in (0,1)$. We say that a system of coordinates $(u^1, \ldots, u^n)$ defined on an open set $U \subset M$ is *harmonic* if $\Delta u^a = 0$ for $1 \leq a \leq n$. Elliptic regularity implies that each coordinate function u^a is of class $C^{k+1,\alpha}$.

Harmonic coordinates are well-suited to the study of regularity problems, as suggested by the following lemma.

Lemma 73. *Let T be a tensor field on M. Suppose g and T are both $C^{k,\alpha}$ with respect* *to some coordinate chart. Then T is $C^{k,\alpha}$ with respect to g-harmonic coordinates.*

This says that any tensor T is at least as regular in harmonic coordinates as it is in an arbitrary coordinate chart. Applying this lemma to $T = g$, we find that the metric has optimal regularity when computed in harmonic coordinates.

Recalling the coordinate expression for the Laplacian,

$$\Delta u = g^{ab} \left(\frac{\partial^2 u}{\partial u^a \partial u^b} - \Gamma^c_{ab} \frac{\partial u}{\partial u^c} \right)$$

for any function u, we find in harmonic coordinates that $g^{ab}\Gamma^c_{ab} = 0$ for each c. Using this, it is not hard to see that the Ricci curvature is given by

$$-2\,\mathrm{Ric}_{ab} = \Delta(g_{ab}) + Q_{ab}(g^{-1}, \partial g), \tag{A.1}$$

where Q_{ab} is some smooth, universal function that depends quadratically on the inverse and first derivatives of the metric tensor. To avoid confusion, we emphasize that $\Delta(g_{ab})$ denotes the value of the Laplacian acting on the scalar function g_{ab}, rather than the ab-component of the tensor Δg, which is of course zero.

It follows from equation (A.1) that the regularity of g is entirely determined by the regularity of the Ricci curvature tensor, rather than the full Riemann tensor.

Theorem 74. *(DeTurck and Kazdan (1981)) Let g be a C^2 Riemannian metric. If* $\mathrm{Ric}(g)$ *is $C^{k,\alpha}$ (resp. real analytic) in harmonic coordinates for some $k \geq 0$, then in these coordinates g is $C^{k+2,\alpha}$ (resp. real analytic).*

To show that this result is not vacuous, we observe that harmonic coordinates always exist locally.

Lemma 75. *Let g be a Riemannian metric of class $C^{k,\alpha}$ for some $k \geq 1$. For any $x \in M$ there exist $C^{k+1,\alpha}$ harmonic coordinates defined in some neighborhood U containing x.*

As an illustration of this result, we observe that Einstein metrics are always real analytic when viewed in harmonic coordinates. To demonstrate the substance of this result, we consider the example $g = \varphi^* g_0$, where g_0 is the flat metric on $\mathbb{T}^n$, and φ is a diffeomorphism only of class C^3. By construction this metric has $\mathrm{Ric}(g) = 0$, but is only of class C^2 with respect to an arbitrary smooth coordinate chart on M. The above result says that in a "good" (viz. harmonic) set of coordinates, g is in fact real analytic.

In the following section we will consider sequences of Riemannian metrics. There it will be necessary to find neighborhoods of uniform size on which harmonic coordinates exist. Given a sequence $\{g_i\}$ and a point $x \in M$, the above result proves the existence of neighborhoods $U_i = B_{r_i}(x)$ with the propery that there exist g_i-harmonic coordinates on U_i for each i. There is no guarantee, however, that the radii r_i are bounded away from zero as $i \to \infty$. The following proposition shows that this will be the case if certain geometric bounds are satisfied by the entire sequence.

Proposition 76. *(Anderson (1990)) Let (M^n, g) be a Riemannian manifold with $|\mathrm{Ric}(g)| \leq \lambda$. Then for any $C > 1$ and $\alpha \in (0,1)$, there is a constant $\epsilon_0 = \epsilon_0(\lambda, C, n, \alpha)$ with the following property: given any $x \in M$, there exist harmonic coordinates $(u^1, \ldots, u^n)$ on $B_{\epsilon(x)}(x)$, where $\epsilon(x) \geq \epsilon_0 \operatorname{inj}(x)$, such that:*

- $g_{ab}(x) = \delta_{ab}$;
- $C^{-1} \leq g(y) \leq C$ *for all* $y \in B_{\epsilon(x)}(x)$;
- $\epsilon(x)^{1+\alpha} \|g_{ab}(y)\|_{C^{1,\alpha}} \leq C$ *for all* $y \in B_{\epsilon(x)}(x)$.

Thus given a sequence $\{g_i\}$ with $|\mathrm{Ric}(g_i)| \leq \lambda$ and $\operatorname{inj}(g_i) \geq \iota$ for all i, there exists a positive constant $r = r(\lambda, \iota, C, n, \alpha)$ such that each g_i admits harmonic coordinates on any geodesic ball of radius r. Moreover, the components of g in any such coordinate chart are uniformly bounded in $C^{1,\alpha}$.

We finally show that the number of harmonic coordinate charts (satisfying the above $C^{1,\alpha}$ bounds) required to cover M is uniformly bounded above if one also has control of the volume. For convenience we define

$$\mathcal{M}(\lambda,\iota,V)=\{(M,g):|\operatorname{Ric}(g)|\leq\lambda,\operatorname{inj}(g)\geq\iota,\ \text{and}\ \operatorname{Vol}(M,g)\leq V\}\,.$$

With this notation in place we can state the final result of this section.

Corollary 77. *For any $C>1$ and $\alpha\in(0,1)$, there exists $N=N(n,C,\alpha,\lambda,\iota,V)$ such that any $(M^n,g)\in\mathcal{M}(\lambda,\iota,V)$ has a covering $U_1,\ldots,U_N$ with the property that each U_i admits harmonic coordinates satisying the bounds stated in Proposition 76.*

To prove this we require the following covering lemma (see Evans and Gariepy (1992) for details).

Lemma 78. *(Vitali covering lemma) Let $\mathcal{F}$ be a collection of closed balls in M with* $\sup\{\operatorname{diam}(B):B\in\mathcal{F}\}<\infty$*. Then there exists a countable family $\mathcal{G}\subset\mathcal{F}$ of disjoint balls such that*

$$\bigcup_{B\in\mathcal{F}}B\subset\bigcup_{B\in\mathcal{G}}5B,$$

where $5B$ denotes the closed ball having the same center as B and five times the diameter.

Proof. (of Corollary 77) We choose $r=r(\lambda,\iota,C,n,\alpha)$ as given by Proposition 76, and consider the family of balls

$$\mathcal{F}=\{B_{r/5}(x):x\in M\}$$

which obviously covers M. Then the Vitali covering lemma says there is a countable set of points $\{x_i\}$ such that the balls $B_{r/5}(x_i)$ are disjoint, and

$$\bigcup_i B_r(x_i)=M.$$

From the Bishop–Gromov volume comparison theorem (see, for instance, Besse (1987)) we have $\mathrm{Vol}(B_{r/5}(x_i)) \geq \alpha > 0$ for each i, where α depends on r, n and λ. The disjointness condition then implies

$$\sum_i \alpha \leq \sum_i \mathrm{Vol}(B_{r/5}(x_i)) \leq V,$$

so we conclude that the covering $\{B_r(x_i)\}$ is finite, and contains no more than V/α elements. We thus set $N = V/\alpha$ and the proof follows. □

A.2 Convergence theory

The main result in Anderson's convergence theory for Riemannian manifolds with bounded Ricci curvature is the following.

Theorem 79. *(Anderson (1990)) The space of compact Riemannian n-manifolds such that*

$$|\,\mathrm{Ric}\,| \leq \lambda,$$
$$\mathrm{inj} \geq \iota > 0,$$
$$\mathrm{Vol} \leq V,$$

is precompact in the $C^{1,\alpha}$ topology.

Specifically, this means that for any sequence $\{(M_i, g_i)\}$ in $\mathcal{M}(\lambda, \iota, V)$, there exists a compact Riemannian manifold (M, g), a subsequence $\{(M_i, g_i)\}$, and diffeomorphisms $\varphi_i : M \to M_i$ such that $\varphi_i^* g_i \to g$ in $C^{1,\alpha}(M)$. This in particular implies that the manifolds M_i in the convergent subsequence are diffeomorphic, hence there exists only a finite number of diffeomorphism types of n-manifolds in $\mathcal{M}(\lambda, \iota, V)$ for each dimension n.

Since each M_i can be covered by a bounded number of harmonic coordinate neighborhoods with $C^{1,\alpha}$ bounds on the metric (by Corollary 77), there exist embeddings

$$F_i : M_i \to \mathbb{R}^{nN+n}$$

for all i, as in the proof of the Whitney embedding theorem (see Theorem 2.17 in Spivak (2005) for details). One can then apply the Arzelà–Ascoli theorem locally to show that the $F_i(M_i)$ converge a manifold $M \subset \mathbb{R}^{nN+n}$ in $C^{2,\beta}$ for some $\beta < \alpha$.

Appendix B

Weighted Hölder spaces and asymptotically flat manifolds

.

In this appendix we review some basic features of analysis on asymptotically flat manifolds, using the theory of weighted Hölder spaces. We follow the presentation of Chaljub-Simon and Choquet-Bruhat (1979) and Lee and Parker (1987).

Throughout we let (M, g_0) be a smooth Riemannian manifold. All covariant derivatives, integrals, norms, etc. are to be computed with respect to the metric g_0.

B.1 Weighted spaces

We first define a weighting function, σ, in terms of the geodesic distance from a fixed point in M. For a fixed point $x_0 \in M$, we set $\sigma(x) = (1 + d(x_0, x)^2)^{1/2}$, where d is the distance function on M induced by g_0. Then for each integer $k \geq 0$ and $\delta \in \mathbb{R}$, we let $C^k_\delta(M)$ denote the space of functions u of class C^k such that the norm

$$\|u\|_{C^k_\delta(M)} := \sum_{i=0}^{k} \sup_{x \in M} \sigma(x)^{\delta+i} |\nabla^i u(x)|$$

is finite. For instance, $u \in C^0_\delta(M)$ if and only if

$$|u(x)| \leq \frac{C}{d(x_0, x)^\delta}$$

for some positive constant C, so we can think of δ as specifying the rate of decay of u at infinity. (Note that the opposite sign convention is chosen in both Bartnik (1986) and Lee and Parker (1987), where δ describes the rate of growth at infinity).

Next we define the weighted Hölder space $C^{k,\alpha}_\delta(M)$—for k and δ as above, and $0 < \alpha \leq 1$—to be the space of functions $u \in C^k_\delta(M)$ such that the norm

$$\|u\|_{C^{k,\alpha}_\delta(M)} := \|u\|_{C^k_\delta(M)} + \sup_{d(x,y)<\mathrm{inj}(x)} [\min(\sigma(x), \sigma(y))]^{\delta+k+\alpha} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{d(x,y)^\alpha}$$

is finite.

In this definition we are taking the supremum over all x, y such that y is contained in a normal coordinate neighborhood centered at x, and $\nabla^k u(y)$ refers to the tensor at x obtained by parallel transport along the radial geodesic from x to y.

We next list some basic properties of the weighted Hölder spaces.

Lemma 80. *Suppose $k \geq 0$ and $0 < \alpha \leq 1$. The following results hold.*

1. *The spaces $C^k_\delta(M)$ and $C^{k,\alpha}_\delta(M)$ are Banach spaces.*

2. *If $k \leq l$ then multiplication*

$$C^{k,\alpha}_\delta(M) \times C^{l,\alpha}_\epsilon(M) \to C^{k,\alpha}_{\delta+\epsilon}(M)$$

is continuous.

3. *If $k \leq l$, $\alpha \leq \beta$ and $\delta \leq \epsilon$, then the inclusion*

$$C^{l,\beta}_\epsilon(M) \subset C^{k,\alpha}_\delta(M)$$

is bounded.

4. *If $\alpha < \beta$ and $\delta < \epsilon$, and (M, g_0) is complete, then the inclusion*

$$C^{k,\beta}_\epsilon(M) \subset C^{k,\alpha}_\delta(M)$$

is compact.

B.2 Asymptotically Flat Manifolds

We can now use the weighted Hölder spaces to define the notion of an asymptotically flat manifold. First, we say a Riemannian manifold (M^n, g_0) is *Euclidean at infinity* if there exists a compact set $K \subset M$ such that:

- $M \setminus K$ has a finite number of connected components $M_1, \ldots, M_N$;
- there exist diffeomorphisms $\varphi_i : M_i \to \mathbb{R}^n \setminus B_1(0)$ for $1 \leq i \leq N$;
- $\varphi_i^* g_E = g_0$ on M_i for $1 \leq i \leq n$.

In the last line, g_E denotes the flat metric on $\mathbb{R}^n$. The components $M_1, \ldots, M_N$ are called the *ends* of M.

This is really a topological property—given a smooth manifold satisfying the first two properties in the above definition, one can always construct a smooth, complete Riemannian metric g_0 satisfying the third. We will often refer to an *asymptotically Euclidean manifold* M without explicit reference to the background metric g_0.

A Riemannian metric g on an asymptotically Euclidean manifold M is then called *asymptotically flat of order* τ if $g - g_0 \in C^{1,\alpha}_\tau(M)$. We are primarily interested in the case $\tau > (n-2)/2$, as this is the regime in which the positive mass theorem holds.

Before stating this theorem, we must define the mass of an asymptotically flat manifold. Let (M^n, g) be asymptotically flat, and for convenience assume that M has only one end. Then there exist coordinates $\{x^i\}$ on $M \setminus K$ such that $g_{ij} \to \delta_{ij}$ as $r = \sqrt{\sum (x^i)^2} \to \infty$. Following the notation of Schoen (1989), we define the *ADM mass* to be

$$m_{\mathrm{ADM}}(g) := c_n \lim_{R \to \infty} \int_{S_R} \sum_{i,j} \left(g_{ij,i} - g_{ii,j}\right) \nu^j \, d\mu,$$

where ν and $d\mu$ denote the outward unit normal and volume element, respectively, of S_R computed with respect to g, and c_n is a positive constant that only depends on the dimension.

It was shown by Bartnik (1986) (*cf.* Lee and Parker (1987)) that $m_{\text{ADM}}(g)$ is independent of the choice of asymptotic coordinates through which it was defined—and hence only depends on the metric g—provided $\tau > (n-2)/2$ and $R(g) \in L^1$.

We conclude this section with the statement of the positive mass theorem.

Theorem 81. *(Schoen and Yau (1979)) Let (M^n, g) be asymptotically flat of order $\tau > (n-2)/2$. (If $n > 7$, assume additionally that M is a spin manifold.) If $R(g) \in L^1$ and $R(g) \geq 0$, then $m_{ADM}(g) \geq 0$. Moreover, $m_{ADM}(g) = 0$ if and only if (M^n, g) is isometric to $\mathbb{R}^n$ with the flat metric.*

B.3 Elliptic theory

We finally discuss the elliptic theory needed for the proof of Theorem 58. All results in this section are taken from Lee and Parker (1987) and Chaljub-Simon and Choquet-Bruhat (1979).

Proposition 82. *Let (M^n, g) be asymptotically flat of order $\tau > 0$.*

1. *If $u \in C^0_\beta(M)$ and $\Delta u \in C^{0,\alpha}_{\beta+2}(M)$, then $u \in C^{2,\alpha}_\beta(M)$ and*

$$\|u\|_{C^{2,\alpha}_\beta} \leq C\left(\|\Delta u\|_{C^{0,\alpha}_{\beta+2}} + \|u\|_{C^0_\beta}\right).$$

2. *If $0 < \beta < n-2$, $h \in C^{0,\alpha}_\delta(M)$ for some $\delta > 2$, and the operator $\Delta + h : C^{2,\alpha}_\beta(M) \to C^{0,\alpha}_{\beta+2}(M)$ is injective, then it is an isomorphism.*

From this proposition easily follows the existence of harmonic coordinates for asymptotically flat manifolds.

Theorem 83. *Let (M^n, g) be asymptotically flat of order $\tau > (n-2)/2$, and let $x^1, \ldots, x^n$ be asymptotic coordinates on $M \setminus K$. Then there exist functions $u^1, \ldots, u^n$*

on M satisfying $\Delta u^a = 0$ for $1 \leq a \leq n$, and

$$u^a - x^a \in C^{2,\alpha}_{\tau-1}(M) \text{ if } n \geq 4,$$

$$u^a - x^a \in C^{2,\alpha}_{\tau-1-\epsilon}(M) \text{ if } n = 3.$$

We comment briefly on the case $n = 3$. The idea of the proof is to solve the equation $\Delta y^a = -\Delta x^a$ for $y^a \in C^{2,\alpha}_{\tau-1}(M)$ and then set $u^a = x^a + y^a$. This is done by applying Proposition 82 with weight

$$\beta = \tau - 1 > \frac{n-2}{2} - 1 = \frac{n-4}{2}.$$

However, it only follows that $\beta > 0$ for $n \geq 4$, so Proposition 82 does not necessarily apply to the case $n = 3$. This can be overcome by instead solving the equation in a weighted Sobolev space, and then applying a suitable weighted embedding theorem to show that the solution is contained in $C^{2,\alpha}_{\tau-1-\epsilon}(M)$ for any positive ϵ. Details can be found in Lee and Parker (1987).

We finally use harmonic coordinates to give a convenient expression for the ADM mass of an asymptotically flat manifold.

Lemma 84. *Let $(u^1, \ldots, u^n)$ be harmonic functions on M, with $u^a - x^a \in C^{2,\alpha}_\delta(M)$ for some $\delta > (n-2)/2$. Then*

$$m_{ADM}(g) = -c_n \left(\lim_{R\to\infty} \sum_{a=1}^{n} \int_{S_R} \langle \nabla_N du^a, du^a \rangle \right).$$

Proof. The decay condition $u^a - x^a \in C^{2,\alpha}_\delta$ implies that $\{u^a\}$ constitutes a harmonic coordinate system outside a sufficiently large compact set, where we have

$$\left| g^{ab} - \delta^{ab} \right| \leq \frac{C}{r^\delta},$$

$$\left| g_{ab,c} \right| \leq \frac{C}{r^{1+\delta}}$$

for some $C > 0$, with $r = \sqrt{\sum (x^a)^2}$ as before. We thus compute

$$\begin{aligned}
\Gamma^b_{aa} - \Gamma^a_{ab} &= \frac{g^{bc}}{2}\left(2g_{ac,a} - g_{aa,c}\right) - \frac{g^{ac}}{2}\left(g_{ac,b} + g_{bc,a} - g_{ab,c}\right) \\
&= \frac{\delta^{bc}}{2}\left(2g_{ac,a} - g_{aa,c}\right) - \frac{\delta^{ac}}{2}\left(g_{ac,b} + g_{bc,a} - g_{ab,c}\right) + \mathcal{O}(r^{-2\delta-1}) \\
&= g_{ab,a} - g_{aa,b} + \mathcal{O}(r^{-2\delta-1}).
\end{aligned}$$

Since the u^a are harmonic we have $g^{ac}\Gamma^b_{ac} = 0$ for each b, hence

$$\begin{aligned}
\sum_a \Gamma^b_{aa} &= g^{ac}\Gamma^b_{ac} + \mathcal{O}(r^{-2\delta-1}) \\
&= \mathcal{O}(r^{-2\delta-1}).
\end{aligned}$$

We then recall that $N = -\nu$, and write the ADM mass integrand as

$$\begin{aligned}
c_n \sum_{a,b}\left(g_{ab,a} - g_{aa,b}\right)\nu^b &= -c_n \sum_{a,b}\left(\Gamma^b_{aa} - \Gamma^a_{ab}\right)N^b + \mathcal{O}(r^{-2\delta-1}) \\
&= c_n \sum_{a,b} N^b \Gamma^a_{ab} + \mathcal{O}(r^{-2\delta-1}).
\end{aligned}$$

From the definition of the Christoffel symbols we have

$$\begin{aligned}
\langle \nabla_N du^a, du^a \rangle &= -\sum_b N^b \langle \Gamma^a_{bc} du^c, du^a \rangle \\
&= -\sum_b N^b g^{ac}\Gamma^a_{bc} \\
&= -\sum_{a,b} N^b \Gamma^a_{ab} + \mathcal{O}(r^{-2\delta-1}).
\end{aligned}$$

The result follows from the observation that $\int_{S_R} r^{-2\delta-1} \to 0$ as $R \to \infty$ precisely when $\delta > (n-2)/2$. □

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Biography

Graham Hugh Cox was born November 16, 1983, in Salmon Arm, British Columbia, Canada. He is well aware that Salmon—members of the class *Actinopterygii*—do not have arms, but is known on occasion to feign amusement when this is brought to his attention.

He received a Combined Honours BSc in Mathematics and Physics from the University of Victoria in 2006, and an MA and PhD from Duke University in 2008 and 2011, respectively.

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