ON CALIBRATIONS FOR AREA MINIMIZING CONES

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Abstract. Since the Simons cone was proposed by Simons in 1968, many efforts had been made to study the area minimizing hypersurfaces with singularities. In this paper, the original proof of the minimality of Simons cone by Bombieri, De Giorgi and Giusti is revisited. We reduce the problem to a first order differential equation and establish a foliation of minimal hypersurfaces that admits the area-minimizing cone as a level set. We show the existence of solutions through computer simulations.

Contents

1. Introduction

The regularity of area minimizing hypersurfaces in $\mathbb{R}^n$ with $n \leq 7$ was established by Simons in 1968 [10]. Meanwhile, he proved that a series of cones in $\mathbb{R}^{2p}$, namely

$$C_{p,p} = \Big\{ (x, y) \in \mathbb{R}^p \times \mathbb{R}^p \ : \ |x| = |y| \Big\}$$

are locally stable. In 1969, Bombieri, De Giorgi and Gusti proved these cones are actually a global minima of the area functional [2].

The minimality of Simons cone is very crucial to the development of regularity theory of minimal surfaces. Not only provides it the first counterexample to the regularity of minimal hypersurfaces in dimensions larger than 7, but it also completes the proof for Bernstein problem, which states that there exists complete minimal graphs $f : \mathbb{R}^{n-1} \to \mathbb{R}$ which are not hyperplanes when $n \geq 9$. Subsequently, more precise limitations to the singular set of minimal surfaces were given: an $(n-1)$-dimensional minimal hypersurface in $\mathbb{R}^n$ is regular outside a singular set whose dimension is at most $n-8$. This result affects many other related topics, for instance, the regularity of isoperimetric surfaces. The famous proof of Positive Mass Theorem by Schoen and Yau is also limited to dimension less than or equal to 8 because of the regularity problem of minimal hypersurfaces.

It was found later that the Simons cones are actually belong to a larger collection of minimal cones. In $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$, consider

$$C_{p,q} = \left\{ (x,y) \in \mathbb{R}^p \times \mathbb{R}^q \ : \ \frac{|x|^2}{p-1} = \frac{|y|^2}{q-1} \right\}$$

We have the following theorem:

Theorem A. $C_{p,q}$ *is globally area-minimizing if any of the following condition is met:*

(i) *if* $p+q > 8$ *and* $p,q \geq 2$
(ii) *if* $p+q = 8$ *and* $p,q > 2$

The collection of all $C_{p,q}$ of minimal area is called Lawson's cones.

After the first proof of the minimality of Simons cone, many simplification and generalizations were made. These Simons cones were made by different authors. In particular, we would like to mention the work of Lawson [6], Simoes [9], Miranda [7], Davini [3], Morgan [8], Philippis and Paolini [5].

Part (i) of Theorem A was proved by Lawson through the same technique that was used in Bombieri, De Giorgi and Giusti's paper. In 1973, Simoes proved part (ii) and thus complete the proof of Theorem A. During 1980s and 1990s, Miranda provided several simplifications with the aid of computer programs. An elegant and simple proof of the minimality of Simons cone was given by De Philippis and Paolini in 2009 using subcalibration method. The proof was generalized to all but a few exceptional cases of Lawson's cones by De Philippis and Maggi [4] in 2014.

We reinterpreted the proof of Bombieri, De Giorgi and Giusti based on the framework of calibrations. We will reduce the problem to lower dimensions and will eventually study an ordinary differential equation. Using computer program we can show the existence of solution and visualize the solution of the ODE, although the entire proof can be carried out by hand.

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2. Preliminary results

For compact surfaces, the areas are finite and we can compare the total area and determine which surface is minimal. But when it comes to non-compact surfaces like Simons cone, what do we mean by area-minimizing?

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ is a continuous nonnegative function. Consider the following functional:

$$\mathcal{A}\,(\Sigma; B) = \int_{\Sigma\cap B} f \; dA \tag{1}$$

where B is a Borel set in Ω, Σ is in the collection of all oriented smooth hypersurfaces $\mathcal{M}$ and dA is the $n-1$ dimensional volume form. When $f = 1$, this is the standard area functional, denoted as $A(\Sigma; B)$.

We say that Σ is *minimal* (or $\mathcal{A}$*-minimal*) in $\mathcal{M}$ if $\mathcal{A}\,(\Sigma; \overline{U}) \leq \mathcal{A}\,(\Sigma'; \overline{U})$ for any $\Sigma' \in \mathcal{M}$ that is in the homology class of Σ and any precompact open set $U \subset \Omega$.

So for non-compact hypersurfaces, area-minimizing means that it minimizes area when taking intersection with any precompact open set.

It is clear that if Σ is minimal among all hypersurfaces in $\mathcal{M}$, it is a critical point of the functional $\mathcal{A}$, meaning that the first variation of functional $\mathcal{A}$ at Σ is null. Suppose we have a one-parameter family of hypersurfaces $\{\Sigma_t\}_{t\in\mathbb{R}} \subset \mathcal{M}$, with $\Sigma_0 = \Sigma$. The minimality of Σ implies that, for any precompact set U,

$$\frac{d}{dt}\mathcal{A}\,(\Sigma_t; U)\bigg|_{t=0} = 0 \tag{2}$$

We say that a hypersurface Σ is $\mathcal{A}$*-critical* (or simply, *critical*) if it satisfies equation 2 for any one-parameter family Σ_t.

Notice that the collection of smooth hypersurfaces $\mathcal{M}$ does not contain any of the Lawson's cones. Hence, we need to expend the class of our admissible "surfaces". To that end, we need some tools from geometric measure theory. Given a measurable set $E \subset \mathbb{R}^n$, we define *perimeter* of E in $\Omega \subset \mathbb{R}^n$ as[1]:

$$P(E, \Omega) = \sup \left\{ \int_E \operatorname{div} g \, dx \ : \ g \in C_c^1(\Omega, \mathbb{R}), \ |g|_{L^\infty} \leq 1 \right\}$$

We say that a set E has *locally finite perimeter*, if $P(E, \Omega)$ is finite for all bounded open set Ω.

For any measurable set E with locally finite perimeter, we include its boundary ∂E to our collection of "hypersurfaces". To have the functional $\mathcal{A}(\Sigma; \Omega)$ well defined on this enlarged set of hypersurfaces, we will also need to change the volume form to $n-1$ dimensional Hausdorff measure $d\mathcal{H}^{n-1}$ and take Lebesgue integral. We won't go into details on this theory, but everything we defined before is also well-defined in this setting. In fact, the boundary of a measurable set with locally finite perimeter agrees with an oriented smooth hypersurface except on a $\mathcal{H}^{n-1}$ negligible set. The theory of sets with locally finite perimeter can be found in [5].

Moreover, we say that a hypersurface Σ (the boundary of some set E) is minimal if $\mathcal{A}(\Sigma; \overline{U}) \leq \mathcal{A}(\Sigma'; \overline{U})$ for any hypersurface Σ' which is the boundary of some set E' that only disagrees with E in some precompact open set $U \subset \Omega$. Observe that Σ agrees with Σ' outside U and their union is the boundary of the symmetric difference $E \Delta E'$. So in some sense, we can say they are "homologous" to each other.

Equation 2 still holds if Σ is minimal. However, equation 2 is not enough to show the minimality of a critical hypersurface Σ. So the question comes naturally: given a critical hypersurface Σ of functional $\mathcal{A}$, how can we show that it is minimal? One way to show a submanifold minimizes volume in its homology class is through calibration.

3. Calibration methods

A degree p *calibration* of an oriented Riemannian manifold (M, g) is a differential p-form ω such that:

(i) ω is closed, that is, $d\omega = 0$.

[1] $C_c^1(\Omega, \mathbb{R})$ is the set of continuously differentiable functions with compact support in Ω

(*ii*) for any $o \in M$ and any p-dimensional subspace $V \subset T_o M$, the inequality $\omega|_V \leq dV|_V$ always holds, where dV is the volume form respect to g.

We say $\Sigma \subset M$ is a *calibrated submanifold* if the calibration ω equals the induced volume form when restrict to any tangent space of Σ. It is well known through a one line argument that a calibrated submanifold Σ minimizes volume in its homology class:

$$V(\Sigma) = \int_\Sigma \omega = \int_{\tilde{\Sigma}} \omega \leq \int_{\tilde{\Sigma}} dV = V(\tilde{\Sigma})$$

where $\tilde{\Sigma}$ homologous to Σ. The second equality comes from Stroke's theorem since ω is closed and the inequality is obtained through the definition of calibration.

Now in our case, we have $\Omega \in \mathbb{R}^n$ equipped with the standard metric. To calibrate a smooth hypersurface Σ, we need to find a $n-1$ differential form on Ω. However, by identifying the space of vector fields and $n-1$ differential forms on Ω, it is sufficient to exhibit a vector field ξ such that:

(*i*) $\xi = \nu$ on Σ;

(*ii*) $\operatorname{div}(\xi) = 0$ on Ω;

(*iii*) $|\xi| \leq 1$ on Ω,

where ν is unit normal vector field on Σ. If Σ is minimal of the area functional, we have a similar one line argument as before,

$$A(\Sigma; U) = \int_{\Sigma \cap U} \xi \cdot \nu \, d\mathcal{H}^{n-1} = \int_{\Sigma' \cap U} \xi \cdot \nu' \, d\mathcal{H}^{n-1} \leq A(\Sigma'; U)$$

where Σ' is a hypersurface in the homology class of Σ, ν' is the unit normal of Σ' and U is any precompact open set in Ω.

Note that to calibrate Σ with respect to functional $\mathcal{A}$ instead of the standard area functional, we simply replace condition (ii) with $\operatorname{div}(f \cdot \xi) = 0$.

For the boundary of sets with locally finite perimeter, the barrier is to provide a divergence theorem that can compare two "homologous" surfaces. We will employ the following refined version of the divergence theorem:

Lemma 1. *Let ξ be a bounded vector field on Ω. Suppose there exist closed sets S and S_0, where $\mathcal{H}^{n-1}(S_0) = 0$ and $S \setminus S_0$ is a smooth surface and that*

(*i*) *ξ is C^1 and $\operatorname{div} \xi = 0$ on $\Omega \setminus S$*

(*ii*) *ξ is continuous on $\Omega \setminus S_0$*

Then $\operatorname{div} \xi = 0$ *holds distributionally on* Ω*. Morevover, if* E *and* F *are two sets of locally finite perimeters that only disagrees on a precompact open set* $U \subset \Omega$*, then*

$$\int_{\partial E \cap \overline{U}} \xi \cdot \nu_{\partial E} \, d\mathcal{H}^{n-1} = \int_{\partial F \cap \overline{U}} \xi \cdot \nu_{\partial F} \, d\mathcal{H}^{n-1}$$

This divergence theorem allows us to calibrate surfaces that are not necessarily regular. We refer the proof to lemma 2.4 and 2.6 of [1].

Although calibration is a powerful tool, it is not an easy task to find a calibration for a given hypersurface, as there is no standard technique. Fortunately, there are ways to work around.

Sub-calibration method uses the same idea but weakens the condition (ii) on divergence. Suppose $\Sigma = \partial R$ of some region R, sub-calibration only requires $\operatorname{div}(\xi) \leq 0$ on $R \cap \Omega$. Intuitively, such vector field calibrates the hypersurface from one "side", and we can use another vector field to calibrate the surface from the other "side". An elegant and detailed discussion of subcalibration can be found in De Philippis' paper. They use a explicit function $f(x, y) = (|x|^4 - |y|^4)/4$ to sub-calibrate the Simons cone.

Another way to find a calibration is through foliation. We say that a collection of hypersurfaces $\{\Sigma_t\}$ is a *foliation* of an open set $\Omega \in \mathbb{R}^n$ if they are pairwise disjoint and $\Omega = \bigcup_t (\Sigma_t \cap \Omega)$. The idea is to form a foliation using level sets of certain real valued function f. Once we have a foliation in which each hypersurface is critical to the area functional, we would want to use the unit normal vectors as the vector field for calibration. One thing we need to make sure is that the divergence has to vanish. This is accomplished by the following proposition.

Proposition 2. *Let* V *be an open set of* Ω *and* $f \in C^2(V, \mathbb{R})$ *such that* $|\nabla f| \neq 0$ *on* V*. Let* $\{\Sigma_t\}$ *be a family of hypersurfaces in* $\mathbb{R}^n$ *such that* $\Sigma_t \cap V = f^{-1}(t)$*. If* Σ_t *are critical for the area functional in* Ω*, then*

$$\operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) = 0 \quad \textit{on } V$$

A proof is given in [3], but we will provide a simpler proof here.

Proof. Fix $x_0 \in V$. Without the loss of generality, we can assume Σ_0 is the unique hypersurface that passes through x_0.

Since $|\nabla f(x_0)| \neq 0$, the inverse function theorem implies that Σ_0 is smooth in an open neighborhood near x_0. So there exists $\varepsilon > 0$ such that $\Sigma_0 \cap B_\varepsilon(x_0)$ is smooth.

We know that Σ_0 is at critical point of the area functional.Hence, equation 2 is true for $U = B_\varepsilon(x_0)$. We then have

$$\frac{d}{dt}A(\Sigma_t; B_\varepsilon(x_0))\Big|_{t=0} = 0$$

By the first variation of the area formula [2], we obtain that

$$A'(\Sigma_0; B_\varepsilon(x_0)) = \int_{\Sigma_0 \cap B_\varepsilon(x_0)} \operatorname{div} \nu_0 \, dA = 0$$

where ν_0 is the unit normal on Σ_0. As $|\nabla f| \neq 0$ at x_0, the unit normal on Σ is just the unit vector of the gradient. Since f is twice continuously differentiable $\operatorname{div} \nu_0$ is continuous. The integral remains 0 for any ε less than some ε_0. This implies that $\operatorname{div} \nu_0 = 0$ at x_0. Therefore,

$$\operatorname{div} \left(\frac{\nabla f(x_0)}{|\nabla f(x_0)|} \right) = 0$$

This is true for all $x_0 \in V$, so we have completed the proof. □

4. $SO(p) \times SO(q)$ invariant foliation

The cone $C_{p,q}$ is $G = SO(p) \times SO(q)$ invariant, meaning that G acts on the cone is the cone itself. Thus, we want to have a foliation such that each hypersurface is also invariant under the action of G. We can use the level sets of some G-invariant function $f : \mathbb{R}^n \to \mathbb{R}$. That means $f(G(z)) = f(z)$ for any $z \in \mathbb{R}^n$. Let $z = (x, y)$ with $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$. Then $f(x, y) = f(x', y)$ if $|x| = |x'|$ and same for variable y. Thus, the value of $f(x, y)$ only depends on $|x|$ and $|y|$.

Now let,

$$S_0 = \Big\{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \cong \mathbb{R}^n \ : \ |x| \cdot |y| = 0\Big\}$$

It is then clear that a G-invariant function $f : \mathbb{R}^n \to \mathbb{R}$ is uniquely determined by its value on S_0 and a function $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$.

The following proposition combines the result of lemma 1 and proposition 2 and shows how we are going to calibrate $C_{p,q}$.

Proposition 3. *Let S_0 be the set defined as above and let $S = C_{p,q} \cup S_0$. Assume $f \in C^2(\mathbb{R}^n \setminus S_0, \mathbb{R}^n)$ is a G-invariant function that has nonzero gradient on $\mathbb{R}^n \setminus S$. If $f^{-1}(0) = C_{p,q} \setminus 0$ and $\nabla f / |\nabla f|$ admits a continuous extension on $f^{-1}(0)$ that agrees with the unit normal of the cone, then there exists a vector field ξ that calibrates the level sets of f, and in particular the cone $C_{p,q}$.*

[2] A proof can be found in [11]

Proof. Let $\xi = \nabla f/|\nabla f|$ on $\mathbb{R}^n \setminus S_0$ and $\xi = 0$ on S_0. This vector field is well defined since the gradient of f is nowhere zero. We observe two facts here.

First, ξ is C^1 and $\operatorname{div}\xi = 0$ on $\mathbb{R}^n \setminus S$. The first part is straightforward since f is twice continuously differentiable. To get the second part, we simply apply proposition 3 with f restricted on $\mathbb{R}^n \setminus S$. Second, ξ is continuous on $\mathbb{R}^n \setminus S_0$ because $\nabla f/|\nabla f|$ admits a continuous extension to $S \setminus S_0$.

When $p, q \geq 2$, $\mathcal{H}^{n-1}(S_0) = 0$. Thus, we can apply lemma 2 here. With the arguments of calibration, the proof is then complete. $\square$

So the proof of Theorem A comes down to find a G-invariant function f that satisfies the conditions in proposition 3. Since f is determined by a function $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, we can then reduce our problem to lower dimensions.

5. First order ODE

Now we see that the essential part of proving Theorem A is to finding an appropriate G-invariant function f. More specifically, we need

$$\operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) = 0 \quad \text{on} \quad \mathbb{R}^n \setminus S_0$$

Since $f : \mathbb{R}^n \to \mathbb{R}$ is G-invariant, we can define $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $F(u, v) = f(x, y)$ where $x = (x_1, ..., x_p) \in \mathbb{R}^p$, $u = |x|$ and $y = (y_1, ..., y_q) \in \mathbb{R}^q$, $v = |y|$. For convenience, we will use subscripts for partial derivatives.

We know that,

$$\begin{aligned}
\nabla f &= \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}\vec{x}_i + \sum_{j=1}^{q} \frac{\partial f}{\partial y_j}\vec{y}_j \\
&= \sum_{i=1}^{p} F_u \frac{\partial u}{\partial x_i}\vec{x}_i + \sum_{j=1}^{q} F_v \frac{\partial v}{\partial y_j}\vec{y}_j \\
&= \sum_{i=1}^{p} \frac{F_u \cdot x_i}{u}\vec{x}_i + \sum_{j=1}^{q} \frac{F_v \cdot y_j}{v}\vec{y}_j \\
|\nabla f|^2 &= F_u^2 + F_v^2
\end{aligned}$$

Compute the divergence

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(\frac{\frac{\partial f}{\partial x_i}}{|\nabla f|}\right) &= \frac{\partial}{\partial x_i}\left(\frac{F_u \cdot x_i}{u\sqrt{{F_u}^2+{F_v}^2}}\right) \\
&= \frac{F_u}{u\sqrt{{F_u}^2+{F_v}^2}} + \frac{x_i^2}{u}\cdot\frac{-F_u^3 - F_u{F_v}^2 + uF_{uu}{F_v}^2 - uF_{uv}F_uF_v}{u^2({F_u}^2+{F_v}^2)^{3/2}} \\
\sum_{i=1}^{p}\frac{\partial}{\partial x_i}\left(\frac{\frac{\partial f}{\partial x_i}}{|\nabla f|}\right) &= \frac{pF_u}{u\sqrt{{F_u}^2+{F_v}^2}} + \frac{-{F_u}^3 - F_uF_v^2 + uF_{uu}{F_v}^2 - uF_{uv}F_uF_v}{u({F_u}^2+{F_v}^2)^{3/2}} \\
&= \frac{(p-1)({F_u}^3+F_u{F_v}^2)}{u({F_u}^2+{F_v}^2)^{3/2}} + \frac{F_{uu}{F_v}^2 - F_{uv}F_uF_v}{({F_u}^2+{F_v}^2)^{3/2}} \\
\sum_{j=1}^{q}\frac{\partial}{\partial y_i}\left(\frac{\frac{\partial f}{\partial y_i}}{|\nabla f|}\right) &= \frac{(q-1)({F_v}^3+F_v{F_u}^2)}{v({F_u}^2+{F_v}^2)^{3/2}} + \frac{F_{vv}{F_u}^2 - F_{uv}F_uF_v}{({F_u}^2+{F_v}^2)^{3/2}} \\
\operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) &= \sum_{i=1}^{p}\frac{\partial}{\partial x_i}\left(\frac{\frac{\partial f}{\partial x_i}}{|\nabla f|}\right) + \sum_{j=1}^{q}\frac{\partial}{\partial y_i}\left(\frac{\frac{\partial f}{\partial y_i}}{|\nabla f|}\right)
\end{aligned}$$

Setting the divergence to 0, we then obtain

$$(p-1)\frac{{F_u}^3+F_u{F_v}^2}{u}+(q-1)\frac{{F_v}^3+F_v{F_u}^2}{v}+F_{uu}{F_v}^2+F_{vv}{F_u}^2-2F_{uv}F_uF_v = 0$$

We know that the divergence is 0 on each level set of f. So let's consider the level curves $F(u,v)=C$ for some constant C. We parametrize the curve as $(u(t),v(t))$. Since F is constant on the curve, we know that

$$dF = F_uu' + F_vv' = 0$$

and similar for second derivative:

$$d^2F = F_{uu}(u')^2 + F_{vv}(v')^2 + F_uu'' + F_vv'' + 2F_{uv}u'v' = 0$$

Combine these two with the equation above, we have that

$$u''v' - u'v'' + (q-1)\frac{(u')^3+(v')^2u'}{v} - (p-1)\frac{(v')^3+(u')^2v'}{u} = 0 \tag{3}$$

It is useful to define the angular parameter:

$$\theta = \arctan\frac{v}{u}$$

which is left invariant by homotheties $u \to \lambda u$ and $v \to \lambda v$ for any $\lambda > 0$. Let's further parametrize the curve by θ,

$$\begin{cases} u = e^{w(\theta)} \cos\theta \\ v = e^{w(\theta)} \sin\theta \end{cases}$$

We have,

$$\begin{cases} u' = \dot{w}u - v \\ v' = \dot{w}v + u \end{cases}$$

$$\begin{cases} u'' = (\ddot{w} + \dot{w}^2 - 1)u - 2\dot{w}v \\ v'' = (\ddot{w} + \dot{w}^2 - 1)v + 2\dot{w}u \end{cases}$$

Put these back to equation 3, we obtain an ordinary differential equation:

$$\ddot{w} = (1 + \dot{w}^2)\left((n-1) + \frac{d - (n-2)\cos(2t)}{\sin(2t)}\dot{w}\right)$$

where $d = p - q$. To simplify further, we set $z = \dot{w}$ and thus,

$$\dot{z} = (1 + z^2)\left((n-1) + \frac{d - (n-2)\cos(2t)}{\sin(2t)}z\right) \tag{4}$$

We want to reduce the problem to study this first order ODE. The following lemma tells us when the existence of solution to equation 4 implies that there's a critical foliation that admits cone $C_{p,q}$ as a level set.

First, we fix some notations. Let γ be the ray on $\mathbb{R}^+ \times \mathbb{R}^+$ such that when acted by group G it results in cone $C_{p,q}$. Furthermore, we denote $\theta_0 = \frac{1}{2}\arccos\frac{d}{n-2}$ between ray γ and u-axis.

Lemma 4. *Let $z(\theta)$ be a solution of equation 4 defined on $\theta \in [0, \pi/2] \setminus \{\theta_0\}$ such that $\lim_{\theta\to\theta_0} |z(\theta)| = +\infty$. If the solution curve has no intersection with γ, then there exists a function f that satisfies the conditions in proposition 3.*

Proof. Consider the curve

$$\rho(\theta) = \left(e^{w(\theta)}\cos\theta,\, e^{w(\theta)}\sin\theta\right)$$

defined by the solution $\dot{w}(\theta) = z(\theta)$. This curve is well-defined on $[0, \pi/2] \setminus \{\theta_0\}$. By classical results on the Cauchy problem, the solution $z(\theta)$ is C^1, so the curve is at least C^2.

Now we want to show that the curve $\rho(t)$ tends asymptotically to the ray γ. Consider the distance to the origin $r(\theta) = |\rho(\theta)| = e^{w(\theta)}$.

Since $z(\theta)$ goes to infinity θ approaches θ_0, we have

$$\dot{r}(\theta_0) = \lim_{\theta \to \theta_0} z(\theta) e^{w(\theta)} = +\infty$$

This implies that $\rho(\theta)$ approaches γ when the angle is close to θ_0.

Now that we have $\rho(\theta)$ that tends to γ asymptotically and it has no intersection with γ, it is not so hard to build a function f. Let ρ_- and ρ_+ be the curves that corresponds to $\rho(t)$ for $\theta < \theta_0$ and $\theta > \theta_0$ respectively. Then the homothetics of these two curves and γ will gives us a foliation on $\mathbb{R}^+ \times \mathbb{R}^+$, with each curve corresponds to $F(u, v) = C$ for some C and γ corresponds to $C = 0$. Specifically, we have a function $\tilde{F}(r, \theta) = F(u, v)$ under polar coordinates:

$$\tilde{F}(r, \theta) = \begin{cases} r/r(\theta) & \text{when} \quad 0 \leq \theta < \theta_0 \\ 0 & \text{when} \quad \theta = \theta_0 \\ -r/r(\theta) & \text{when} \quad \theta_0 < \theta \leq \pi/2 \end{cases}$$

Under the action of G, we then obtain the function f. The properties in proposition 3 are satisfies because $\rho(\theta)$ is C^2 and satisfies equation 4. $\square$

So to prove Theorem A, all we need to show is that when p, q satisfy conditions in Theorem A, the differential equation 4 has a solution defined on $[0, \pi/2] \setminus \theta_0$ with desired properties. We have the following lemma.

Lemma 5. *For p, q satisfying either part (i) or part (ii) in Theorem A, there exists a solution $z(t)$ defined on $[0, \theta_0) \cup (\theta_0, \pi/2]$ such that $\lim_{\theta \to \theta_0} |z(\theta)| = +\infty$ and the solution curve does not intersect with γ.*

Bombieri, De Giorgi and Giusti have proved the existence of solutions to a differential equation that is equivalent to equation 4 in the case of Simons cone. Simoes extends the their resuls to all Lawson's cones. Various proofs of lemma 5 (with equivalent differential equations) can be found in [2], [9], [7].

6. Computer simulation results

Here we will use computer program Mathematica to demonstrate the existence of solution to equation 4.

When $p, q \geq 2$ and $n > 8$, the solution will yield a curve that tends asymptotically to the ray γ. Figure 1 show the case when $p = q = 5$.

When the solution exists and $|z(\theta)|$ goes to infinity when the angle approaches the angle of the ray γ, the curve will never intersect with

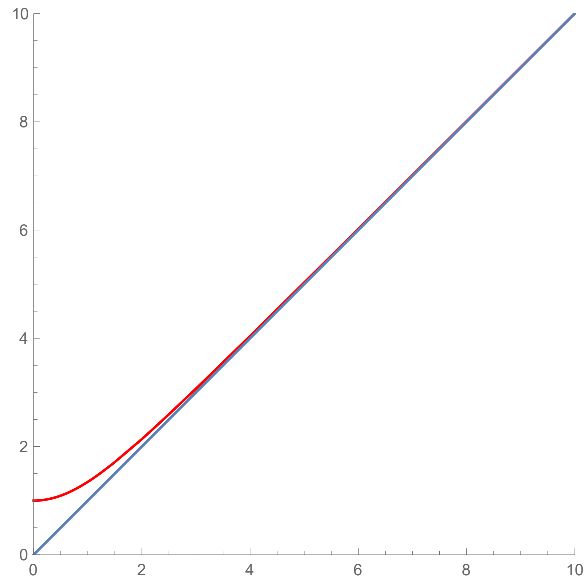

FIGURE 1. Solution to equation 4 when $p = 5$ and $q = 5$

γ. Figure 2 shows that when such curve exists, the homothetics of the curve result in a foliation the calibrates the cone.

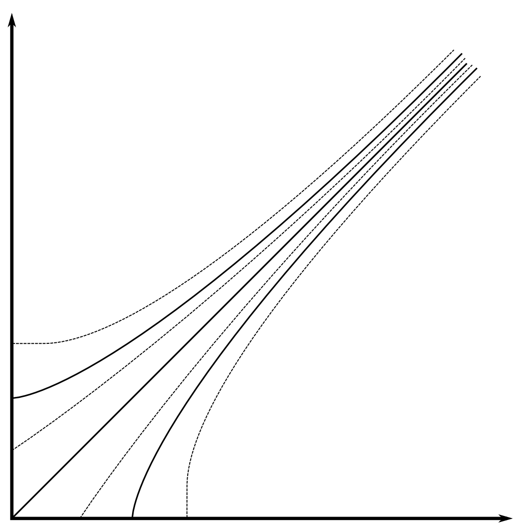

FIGURE 2. Homothetics of the curve forms a foliation that calibrates the cone

However, when p, q does not satisfy the conditions in Theorem A, the solution exists, but the curve will intersect with γ and therefore there exists no foliation that calibrates the cone.

Figure 3 shows that when $p = 2$ and $q = 6$. Though it is not obvious through the picture, there is an intersection between the solution curve and the ray γ.

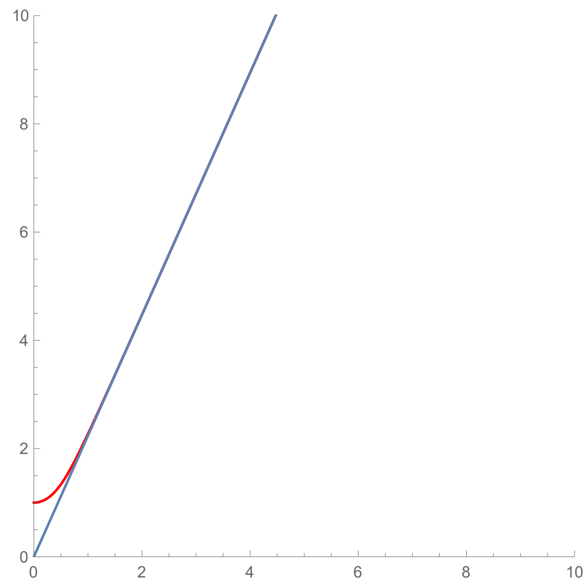

FIGURE 3. Solution when $p = 2$ and $q = 6$

Therefore through computer simulation, we can see that the desired solution only exists when p and q satisfy the conditions in theorem A. Combining the results from previous sections, this demonstrates the minimality of Lawon's cones.

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