Strategy and Effectiveness:
An Analysis of Preferential Ballot Voting Methods

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Abstract:

To provide insight on various voting systems, we study six election methods using three categories of analysis. First, we prove and discuss the various fundamental election properties satisfied by each method. Since the better election methods tend to satisfy more of these properties, we are able to narrow down the list of preferable voting systems. The next phase focuses on the "crowding out" of candidates in elections. We study the susceptibility of each voting system to this crowding phenomenon, verifying that the best methods are those that do not tend to exhibit this problem. Finally, we take two of the best voting systems and run simulated random elections to assess how often they choose the same winner and which method has the best head-to-head winning percentage. We compare these top methods to another system from the original pool of six as a control example. This thesis should help inform studies of how to choose the best election method and provide a recommendation regarding which methods are generically the best.

Contents

1. Acknowledgements

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2. Introduction: Preferential Ballots

2.1 Voter Preferences

Various election methods, such as Single Vote Plurality, have been questioned for the way they extract information from voters. Over the last half-century, political scientists, social scientists, and economists have devised a multitude of election systems that attempt to compile voter preferences efficiently and fairly in order to ensure elected leaders are representative of the voting population. There are many different ways to take voter preferences into account, and each method has its own strengths and weaknesses based on the properties it satisfies.

Arrow's theorem states that there is no perfect voting system that satisfies the following fundamental properties of universality, monotonicity, independence of irrelevant alternatives, citizen sovereignty, non-dictatorship (Hodge, 68). Also called "Arrow's paradox," the theorem is based on the fact that certain conditions preclude the satisfaction of others within the set. Arrow's theorem prompts the question of which system is the best, if not perfect. This inquiry is quite subjective because it requires judgments regarding which of the above properties are most valuable to a system. Naturally, this value judgment differs from person to person and group to group. Furthermore, even if a system is chosen that satisfies four of the five main, desirable conditions, it must be fully comprehended by the voting population and have the feasibility to be implemented without significant cost to the group.

In this paper we will compare preferential ballot systems to the commonly used Single Vote Plurality system and analyze these methods based on their compliance with the properties listed in Arrow's theorem and others such as the Condorcet criterion. Preferential ballots are able to convey more information about voter preferences because they allow voters to rank their candidates in order from most desired to least desired. For this reason, we will focus on the following preferential ballot systems: Instant Runoff Voting, Borda Count, Instant Runoff Borda, Least Worst Defeat, and Ranked Pairs.

2.2 Properties of Effectiveness

2.2.1 Universality

An election system that satisfies universality is one that, given a ranking of candidates in a preferential ballot, can return all possible rankings of candidates. Put differently, universality means there are no restrictions on the outcomes of an election. Additionally, if all voters submit transitive ballots, then the election method must return a transitive ordering of candidates as its output (Hodge, 68). In the modern political landscape, it makes sense for all unique permutations of candidates to be possible rankings; otherwise the election would not adequately incorporate voter preferences.

2.2.2 Monotonicity

An election system is monotone if a candidate's ranking in the outcome of an election is not negatively affected by receiving more votes. In other words, "more votes are better, or at least as good." This condition is important for voting systems as no candidate should be made worse off if they receive additional support from the voting population. If this were not the case, then the system would not be taking into account complete voter preferences for the given candidate.

2.2.3 Independence of Irrelevant Alternatives

Independence of Irrelevant Alternatives (IIA) has two sub-properties. The first states that in any head-to-head matchup between two candidates, A and B, no other candidate in the election pool may affect the outcome of this pair-wise matchup between A and B. This implies the second sub-property that the removal of one or more losing candidates should not affect the outcome of the election. We will also refer to this property as loser independence.

2.2.4 Citizen Sovereignty

Citizen or voter sovereignty is similar to universality in that everyone within the voting population must, without restriction, have a say in the outcome of an election. No prescribed order or condition may be imposed on the system because that would interfere with the incorporation of voter preferences.

2.2.5 Non-dictatorship

An election method is non-dictatorial if all voters' preferences are weighted equally. Put differently, no one voter can have a disproportionate say in the outcome of an election or alter the outcome of an election by themselves.

2.2.6 The Majority Criterion

The majority criterion states that if a candidate gains at least a simple majority of votes (greater than 50%), then that candidate is elected as the winner. A voting system satisfies this property if any candidate that has at least a simple majority of first-place votes is selected as the winner.

2.2.7 The Condorcet Condition

A Condorcet winner is a candidate that wins all head-to-head matchups in an election. Since we will assume there are no ties between candidates, there may only be one Condorcet winner per election, regardless of voting system used. An election system that satisfies the Condorcet winner criterion (Hodge, 40) elects a Condorcet winner if one exists in the candidate pool. Conversely, a Condorcet loser is a candidate that loses all head-to-head matchups with other candidates. The Condorcet winner criterion is an important indicator of the effectiveness and representative nature of an election method, since any election that chooses the undefeated winner is one that takes in a significant amount of information from voter's preferences. Thus, it makes sense for the best election methods to satisfy this condition. While the Condorcet winner criterion is a focal point of several popular preferential ballot systems, it is important to consider election methods that do not allow for Condorcet losers to be elected. A winning candidate who loses all head-to-head matchups would not be very representative of voter preferences, so voting systems that prevent such winners from being elected are more effective. Such methods are said to satisfy the Condorcet loser criterion (Hodge, 40). Methods that satisfy the Condorcet winner criterion are also referred to as Condorcet methods.

2.3 The Crowding Phenomenon

The crowding phenomenon is prevalent in many of today's election systems. The dominance of the two-party system in the United States is an example of candidates being "crowded out" in elections. In these scenarios, a pool of candidates becomes narrowed so that voters are compelled to vote for one of the candidates with the higher perceived chance of winning to avoid the risk of their vote going to waste. This system can feed off of itself when voters focus solely on a narrow field of candidates rather than allowing their true preferences to be seen. Thus, any election system that is susceptible to crowding influences voters to behave in a way that masks their preferences.

2.4 Summary of Approach

In this paper we discuss winning strategies for the last candidate to enter an election based on the established positions of the other candidates. These strategies and the availability of winning positions will inform comparisons among preferential ballot systems and comparisons between these preferential ballot systems and Single Vote Plurality. The analysis is conducted using the unit interval model, a political spectrum of possible candidate positions from zero to one.

The second part of the study focuses on two of the most popular preferential ballot systems, Least Worst Defeat and Ranked Pairs, which satisfy almost all of Arrow's properties and the Condorcet winner condition. We study them in the context of random elections rather than the unit interval model to add breadth to the discussion. This paper compares the properties these methods satisfy and expounds on the winning outcomes they produce for a varying number of candidates. This is done through simulating random elections using MATLAB.

Lastly, we will tie the discussion of properties and simulations together by commenting on the practicality and feasibility of implementation of recommended voting systems. A system is only as effective as the level of understanding of the voters who use it. If an election method is too complicated for voters to understand or for organizations to implement, then it may not be the best method to use for a given group, election, or other circumstance.

3. Assumptions and Notation

3.1 Terms and Notation

To be precise and clear in the following assumptions and analysis, we will define several important terms that are used heavily. First and foremost, an election is a compilation of all voter preferences for a given set of candidates. An election voting system is a function that maps these preferences as inputs to a ranking of candidates as the output. In working with voter preferences, we will assume there are no ties and use the "greater-than" symbol to denote that candidate A received strictly more votes than candidate B (e.g. A>B). We will use the terms "head-to-head" and "pair-wise" to refer to matchups between two candidates within an election.

3.2 Assumptions for Unit Interval Model

The main medium for an analysis of Single Vote Plurality and the preferential ballots is the unit interval model [0, 1]. This research is built on the following fundamental assumptions:

1. Voters are uniformly distributed along unit interval [0, 1] and each voter has one vote.
2. $X_1 \dots X_n$ are candidates that choose unique positions on [0, 1].
3. $X_1 < X_2 < \dots < X_n$, but in general order is arbitrary.
4. $P(X_i)$ represents the percentage of the vote candidate X_i receives.
    - $P(X_i) = X_i + 0.5*(X_{i+1} - X_i)$ for i=1
    - $P(X_i) = 1-X_i + 0.5*(X_i - X_{i-1})$ for i=n
    - $P(X_i) = 0.5*((X_i - X_{i-1}) + (X_{i+1} - X_i))$ for $i \in [2, n-1]$
5. There can only be one winning candidate (no ties).
6. Strategies for positions depend on the infinitesimally small number epsilon ($e \approx 0$):
    - $\text{Lim}_{n\to 0}(e) = 0$
    - Example: Left strategy position ($Z = X - e$)

3.3 Three Candidate Case for Unit Interval Model

The following assumptions provide the foundation for the three-candidate cases, where X and Y are candidates that position themselves before the last candidate Z chooses a location. We use this different notation to make the accompanying graphs easier to understand and fit into this discussion. In generalizing, we consider $(X, Y, Z) = (X_1, X_2, X_3)$, where "n-1" is the number of candidates which position themselves before the last candidate enters. Thus, "n" is the total number of candidates in the election. We refer to candidates and their respective positions on [0, 1] interchangeably.

Three Candidate Assumptions (for n = 2):

1. X and Y are the first two candidates to choose their positions on [0, 1].
2. Y > X → both are interchangeable since order is arbitrary and this case generalizes for Y < X.
    a. To graph winning strategies for Y < X, reflect boundaries and regions about the line Y = X.

3.4 Random Election Method

In order to expand the scope of our analysis, we will use the random election method to analyze some preferential ballots that rely on margin of victory matrices as inputs. A margin of victory matrix contains all the margins of victory of pair-wise matchups between candidates. The rows and columns all represent the candidates, so the margin of victory matrix should have zeros along the diagonals, such as in the following example.

MOV:			
	0	-9.9994	-7.9932
	9.9994	0	-7.9996
	7.9932	7.9996	0

Consider three candidates A, B, and C. Each row in the above matrix corresponds to a candidate, so row one refers to candidate A, row two to candidate B, and row three to candidate C. We use this notation for all cases and examples involving margin of victory matrices. The above outcomes can be re-written in pair-wise form:

- B>A by 9.9994
- C>A by 7.9932
- C>B by 7.9996

We will refer to candidates and their vote totals interchangeably. In general, we have the following properties for each margin of victory (MOV) entry:

- MOV(A, B) = A-B = -MOV(B, A)
- MOV(A, A) = 0

The margin of victory matrix is anti-symmetric, meaning that all entries that are symmetric across the diagonal have opposite signs. In order to simulate random elections, we use a uniformly distributed random number generator to fill in the entries. To virtually eliminate the possibility of ties between candidates and margins of victory, we perturb each entry using another random number, scaled down by a factor of 100, by adding it to entries in the upper triangle of the matrix and subtracting it from elements in the lower half to maintain inverse symmetry. The values of the margins of victory themselves are not as important as their magnitude relative to one another. The margin of victory matrix is then used as an input for some of these preferential ballot elections.

4. Election Methods

4.1 Single Vote Plurality

Single Vote Plurality (SVP) is a common voting system used in many government elections in the U.S. and world-wide. This system grants all voters a single vote for their candidate of choice. These votes are then compiled and the candidate with the highest number or percentage of votes is the winner. This particular method has come under scrutiny because it potentially allows for a candidate to win an election without winning at least 50% of the vote. With more than three or four candidates, it becomes increasingly likely that a candidate could win an election while receiving only a small fraction of the total votes. A candidate who does not win at least a simple majority of the vote does not completely represent voter preferences.

However, Single Vote Plurality satisfies the majority criterion because if a candidate gains a majority of the vote, that candidate would automatically have more votes than any other candidate in the field, thus winning the election. Single Vote Plurality also satisfies the Condorcet loser criterion, since it is impossible for a candidate who loses all head-to-head matchups to gain a plurality of votes. Below is an example of such an election.

Candidate	First Place Votes
A*	**30***
B	20
C	15

*A wins since A has the most first place votes

Although this system satisfies the Condorcet loser criterion, it fails the important Condorcet winner criterion because of the emphasis placed on the magnitude of pair-wise margins of victory. A candidate could win all head-to-head matchups by small margins and lose to another candidate who may lose a matchup, but may have much higher margins of victory. Consider the following example:

MOV:	**0**	**-2**	**100**
	2	**0**	**5**
	-100	-5	0

In this case, candidate two, the candidate corresponding to the second row, is the Condorcet winner. However, candidate one would win the plurality election because he would receive more votes, by virtue of his much large victory over candidate three.

In the special case of two candidates, the plurality system becomes a simple majority vote. With four or more candidates, it becomes much more likely that a winner could be elected without gaining a majority of voter support. Single Vote Plurality does not allow for ranked or preferential ballots, so the information input into the election function is more limited. We use Single Vote Plurality interchangeably with “Plurality” when referring to this method.

Single Vote Plurality properties:

- Universality
- Monotonicity
- Citizen Sovereignty
- Non-dictatorship
- Majority Criterion
- Condorcet Loser Criterion

4.2 Instant Runoff Voting

Instant Runoff Voting (IRV) is a popular preferential ballot method used on many college campuses around the world. As a runoff method, this system eliminates a candidate each round and recalculates first place vote totals each successive round. Voters begin by submitting their ranked preferences for the candidate pool. All first place votes are compiled so that the candidates are ranked by the number of first-place votes they receive. The candidate with the lowest number or percentage of first-place votes is eliminated in the first round. That candidate's name is then removed from all the ballots and the first-place votes are counted up again as if the eliminated candidate were never there in the first place. The next candidate with the lowest number of first-place votes is eliminated in the second round and this process continues again until one candidate remains and is declared the winner. Put differently, each round the submitted ballots that had the eliminated candidate in first place have the rest of the ranked preferences redistributed across the remaining candidates. For example, if candidate A is eliminated and one of the ballots with A as the first choice had B as the second choice, then candidate B would receive that second place vote as a "first place vote" in the next round because A's name would have been wiped off all of the ballots. An example of such an election is given below:

Round 1:

	Number of Votes	**Number of Votes**	**Number of Votes**
Rank	**15**	**10**	**6**
1	A	B	C
2	B	A	B
3	C	C	A

Candidate	**First Place Votes**
A	15
B	10
C*	6*

*C is eliminated in the first round

Round 2:

	Number of Votes	Number of Votes
Rank	**15**	**16***
1	A	B*
2	B	A

*B inherits 6 second-place votes as first place votes because C was eliminated in the last round

Candidate	First Place Votes
B	**16**
A*	15*

*A is eliminated in the second round, so **B wins**

Instant Runoff Voting does not satisfy monotonicity because a candidate receiving a higher ranking in a ballot or set of ballots could result in them getting eliminated sooner. An example of this occurrence is provided below.

Round 1:

	Number of Voters	Number of Voters	Number of Voters	Number of Voters
Ranking	**20**	**15**	**12**	**4**
1	A	B	C	C
2	B	A	B	A
3	C	C	A	B

Candidate	First Place Votes
A	20
C	16
B*	**15***

In this example, candidate B has the fewest first-place votes, so B is eliminated first. After B's name is cleared from all ballots, the preferences in the second and final round are listed below.

Round 2:

	Number of Voters	Number of Voters	Number of Voters	Number of Voters
Ranking	**20**	**15**	**12**	**4**
1	A	A	C	C
2	C	C	A	A

Candidate	First Place Votes
A*	**35***
C	16

Since A has more first-place votes, A is declared the winner. Now, consider a scenario where A receives four more first-place votes by virtue of the last four voters switching the positions of C and A on their ballots.

Round 1:

	Number of Voters	Number of Voters	Number of Voters	Number of Voters
Ranking	**20**	**15**	**12**	**4**
1	A	B	C	A
2	B	A	B	C
3	C	C	A	B

Candidate	First Place Votes
A	24
B	15
C*	**12***

The change in preferences for the last four voters causes C to receive the fewest first-place votes and thus be eliminated.

Round 2:

	Number of Voters	Number of Voters	Number of Voters	Number of Voters
Ranking	**20**	**15**	**12**	**4**
1	A	B	B	A
2	B	A	A	B

Candidate	First Place Votes
B*	**27***
A	24

Since C is eliminated instead of B, B inherits the second place votes from the ballots where C was ranked first, which allows B to amass enough inherited votes to beat A. Since A originally received more first place votes, but finished at a lower rank than before, this election system is not monotone. By nature of being a runoff method of round-by-round eliminations, and as the example above illustrates, Instant runoff also fails the loser independence condition. This makes sense because round-by-round eliminations would be unnecessary if the outcome were to be unchanged. Below is a summary of the conditions satisfied by Instant Runoff Voting.

Instant Runoff Voting properties:

- Universality
- Citizen Sovereignty
- Non-dictatorship
- Majority Criterion
- Condorcet Loser Criterion

4.3 Borda Count

The Borda Count (BC) system is a method typically used in producing national college football and basketball rankings. This system assigns point values to the "ith-place" votes each candidate receives. For example, in a field of n candidates, a first place vote is worth n-1 points, a second place vote is worth n-2 points, and a last place vote garners no points (Hodge, 25). The successive point allocations decrease by one for every step down the ranking scheme. Each candidate's point totals are calculated based on a given point system and their totals are referred to as "Borda scores." The candidate with the highest score is the winner of the Borda Count election. Changing the point value of each place in the ranking can significantly alter the outcome of a Borda election. In the following example, we will use the point system explained by Hodge and Klima (25).

	Number of Voters	**Number of Voters**	**Number of Voters**	**Number of Voters**
Ranking	**30**	**20**	**10**	**5**
1	B	A	C	C
2	A	B	B	A
3	C	C	A	B

	First Place Votes	**Second Place Votes**	**Third Place Votes**
Ranking	**2 points**	**1 point**	**0 points**
A	20	35	10
B	30	30	5
C	15	0	50

Ranking	**Borda Score**
B*	**90***
A	75
C	30

*B is the winner with the highest Borda score

The Borda Count system has the property that the winner is the candidate whose row in the margin of victory matrix has the highest sum of its elements (Wright, 15). This makes sense because the candidate with the highest sum of row elements will have the greatest aggregate margin of victory over all head-to-head matchups with other candidates. Calculating these row sums is akin to adding all of the weighted first, second, and n-th place votes for each candidate and returning the highest point total. Regardless of how many points the last place ranking garners a candidate, each candidate will start off with m points, where m is equal to the number of candidates multiplied by the number of points for last place (Wright, 16). This is because the worst case scenario for any candidate is to be ranked last on all of the ballots. The remaining points each candidate receivers is an increasing function of the number of head-to-head wins the candidate has over others in the pool. Therefore, a ranking of the sums of the elements in the rows of the margin of victory matrix will tell us the final Borda Count outcome ranking of the candidates.

Borda also fails the majority criterion, since it is possible for a candidate to get a majority of first-place votes, but not enough second-place votes to win the election. Consider the example below.

	Number of Voters	**Number of Voters**	**Number of Voters**	**Number of Voters**
Ranking	**21**	**22**	**7**	**37**
1	A	B	C	C
2	B	A	B	A
3	C	C	A	B

	First Place Votes	**Second Place Votes**	**Third Place Votes**
Ranking	**2 points**	**1 point**	**0 points**
A	21	59	7
B	22	28	37
C	44	0	43

Ranking	**Borda Score**
A*	**101***
C	88
B	72

In this example, candidate A wins the election despite having the lowest total of first place votes. Candidate C has the majority of first place votes, but does not win due to a lack of second-place votes. Since Borda fails to satisfy the majority criterion, it also fails the Condorcet winner criterion. Put differently in the contra-positive, the majority winner must be ranked higher on more ballots than each other candidate. The Condorcet winner must also win every head-to-head matchup and thus must be ranked higher on more ballots than each of the other candidates. This means that every majority winner is a Condorcet winner, but not every Condorcet winner is a majority winner. Therefore, if an election does not satisfy the majority criterion, it also does not satisfy the Condorcet winner criterion.

However, Borda Count satisfies the Condorcet loser criterion because it is impossible for a candidate to be ranked lower on more ballots than any of the other ballots and still achieve a high enough Borda score to win. Below is a summary of the Borda Count properties.

Borda Count properties:

- Universality
- Monotonicity
- Citizen Sovereignty
- Non-dictatorship
- Condorcet loser criterion

4.4 Instant Runoff Borda Count

Similar to Instant Runoff Voting, Instant Runoff Borda Count (IRBC) uses a round-by-round elimination system to remove the candidate with the lowest Borda score each round. The Borda scores are then recalculated each round until the winner is the candidate left after all the eliminations. An example of this runoff is shown below.

Round 1:

	Number of Voters	**Number of Voters**	**Number of Voters**	**Number of Voters**
Ranking	**21**	**22**	**7**	**37**
1	A	B	C	C
2	B	A	B	A
3	C	C	A	B

	First Place Votes	**Second Place Votes**	**Third Place Votes**
Ranking	**2 points**	**1 point**	**0 points**
A	21	59	7
B	22	28	37
C	44	0	43

Ranking	**Borda Score**
A	101
C	88
B*	**72***

*B has the lowest Borda score and is eliminated from the ballots

Round 2:

	Number of Voters	**Number of Voters**	**Number of Voters**	**Number of Voters**
Ranking	**21**	**22**	**7**	**37**
1	A	A	C	C
2	C	C	A	A

	First Place Votes	Second Place Votes
Ranking	**1 points**	**0 point**
A	43	44
C	44	43

Ranking	Borda Score
C*	**44***
A	43

*C has the higher Borda score and is declared the winner

Because of the runoff component and the point-based system, this election method fails to satisfy the monotonicity condition and the IIA property. Unlike Borda count, IRBC satisfies the Condorcet winner criterion. The following proof is an adaptation of the one written by Barry Wright (32).

Consider an election with n candidates and q voters (and ballots since each voter gets one ballot), with each i-th place vote garnering n-i points. We now calculate the total number of points in the Borda count election garnered by the vector of candidates $\mathbf{X} = [X_1, \ldots, X_n]$.

$$BC(\mathbf{X}) = q * (0 + 1 + 2 + \ldots + n\text{-}2 + n\text{-}1)$$
$$= q*n*(n-1)/2$$

Using this result, we derive the average Borda Count score, $BC_{avg}(\mathbf{X}) = q*(n-1)/2$. If a Condorcet winner exists (C), he must be ranked higher than each other candidate on more than q/2 ballots. Since n-1 is the minimum number of candidates that must be ranked below him on the voter ballots, we have the following relationship.

$$BC(C) > q*(n-1)/2 = BC_{avg}(\mathbf{X})$$

Since the Borda score of the Condorcet winner must always be greater than the average score across all candidates, C can never be eliminated, and thus must always win the IRBC election. This also implies that IRBC satisfies the majority criterion. Below is a summary of the properties satisfied by Instant Runoff Borda.

Instant Runoff Borda Count properties:

- Universality
- Citizen Sovereignty
- Non-dictatorship
- Majority Criterion
- Condorcet Winner Criterion
- Condorcet Loser Criterion

4.5 Least Worst Defeat

Least Worst Defeat (LWD) is another preferential ballot system that uses a margin of victory matrix to determine a winner. The winner in this election is the candidate with the worst loss that is smallest in magnitude out of all the other candidates. This candidate can be found by searching for the minimum values within each row of the margin of victory matrix, taking the maximum value out of that set of minimum values, and returning that value's row number, which corresponds to the winning candidate. Least Worst Defeat satisfies the Condorcet winner criterion because an undefeated candidate would have a lowest, worst defeat of zero in their corresponding matrix row. With the exclusion of ties, the election function would return this candidate as the winner using the method just described above. This also implies that Least Worst Defeat satisfies the majority criterion. However, Least Worst Defeat fails to satisfy the Condorcet loser condition because the winning candidate could lose all pair-wise matchups and still have the lowest worst defeat, as shown by the following example.

MOV:	0	10	100	-20
	-10	0	-5	-2
	-100	5	0	25
	20	2	-25	0

In the above election, candidate two, represented by the second row, loses to all other candidates head-to-head, but has the lowest worst defeat of -10. This means that candidate two is elected the winner, despite being a Condorcet loser. In the case where there is no Condorcet winner and a Condorcet loser exists, Least Worst Defeat can select the Condorcet loser as a winner if their margins of loss are small enough in magnitude. LWD also satisfies monotonicity, since more votes for a candidate reduce their margins of loss and increase margins of victory so that they cannot be worse off. Least Worst Defeat satisfies the following properties.

Least Worst Defeat properties:

- Universality
- Monotonicity
- Citizen Sovereignty
- Non-dictatorship
- Majority Criterion
- Condorcet Winner Criterion

4.6 Ranked Pairs

Ranked Pairs (RP) is a preferential ballot voting system that also relies on a margin of victory matrix. Ranked Pairs takes all pair-wise matchups with positive margins of victory and ranks them in descending order. To ensure that there are no ties, we use the random number generator addition described earlier to make all margins of victory distinct. The final ranking is assembled by taking each pair-wise matchup in order by rank and enforcing the transitive property of voter preferences. Any matchup that violates the assembled ranking order at any step is ignored as contradictory. Consider the following simple example with three candidates:

Matchup Outcome	**Margin of Victory**
B>C	30
A>B	20
C>A	10

Based on the top two matchup outcomes, the ranking order would be A>B>C by transitivity. The last outcome, C>A, has a margin of victory of lower magnitude than the previous ones, so rather than being included in the final order, it is ignored. Thus, the result of this election is A>B>C as the final ranking with no ties.

Like Least Worst Defeat, Ranked Pairs satisfies the Condorcet winner criterion. Since a Condorcet winner has to win all pair-wise matchups, none of these winning outcomes would be ignored as contradictory. In order for a matchup to be designated as contradictory, the Condorcet winner would have had to lose at some point with a higher margin of loss. However, since the Condorcet winner cannot lose head-to-head, this scenario is impossible. Since Ranked Pairs is a Condorcet method, it satisfies the majority criterion as well. Unlike Least Worst Defeat, however, Ranked Pairs satisfies the Condorcet loser criterion, meaning that a Condorcet loser cannot win a Ranked Pairs election. In order for a candidate to be top-ranked, they would have to win at least one pair-wise matchup. Since a Condorcet loser has no head-to-head wins, this is impossible. A summary of the properties Ranked Pairs satisfies is listed below.

Ranked Pairs properties:

- Universality
- Monotonicity
- Citizen Sovereignty
- Non-dictatorship
- Majority criterion
- Condorcet winner criterion
- Condorcet loser criterion

4.7 Summary of Properties

The following is a summary of the voting methods discussed in this paper and the conditions they satisfy.

Voting Systems	Universality	Monotonicity	IIA	Citizen Sovereignty
SVP	Yes	Yes	No	Yes
IRV	Yes	No	No	Yes
BC	Yes	Yes	No	Yes
IRBC	Yes	No	No	Yes
LWD	Yes	Yes	No	Yes
RP	Yes	Yes	No	Yes

Voting Systems	Non-dictatorship	Majority criterion	Condorcet winner criterion	Condorcet loser criterion
SVP	Yes	Yes	No	Yes
IRV	Yes	Yes	No	Yes
BC	Yes	No	No	No
IRBC	Yes	Yes	Yes	Yes
LWD	Yes	Yes	Yes	No
RP	Yes	Yes	Yes	Yes

It is striking to note that IIA is the only condition that is failed by all of the methods, including non-Condorcet methods, studied. In his thesis, Barry Wright (58) proves that only Approval Voting satisfies this condition. Approval voting allows candidates to, in no particular order, vote for as many candidates as they want to win, and the winner is the candidate with the most approval votes. Since none of the voting systems studied in this paper satisfy loser independence, we will not discuss it more in depth when comparing methods.

Several other trends can be seen from looking at the tables above, which confirm important relationships between some of the properties studied. First, since the Condorcet winner must win all head-to-head matchups, they must also be the majority winner. Thus, satisfaction of the Condorcet winner criterion implies compliance with the majority criterion (Wright,32). The contra-positive of this statement must also be true, meaning that an election system that fails the majority criterion must also fail the Condorcet winner criterion.

Additionally, the Condorcet winner criterion and IIA are incompatible (Wright, 57). This is one component of Arrow's Paradox, which states that the satisfaction of certain properties naturally prevents the satisfaction of one or more of the other properties. Consider the following matrix where A, B, and C are constant margins of victory.

MOV:			
	0	A	-B
	-A	**0**	**C**
	B	-C	0

Suppose that by some method, the second candidate, indicated by the second row. If the third candidate (third row) drops out, the second candidate should still be elected as the winner. However, once this happens, we have the following margin of victory matrix.

MOV:		
	0	A
	-A	0

If the Condorcet condition is to be satisfied, the first candidate must be chosen the winner. However, if IIA is to be satisfied, then the second candidate must remain the winner. Therefore, these two properties cannot be satisfied at the same time by one election method.

Similarly, by their design, instant runoff methods that use rounds of elimination fail the loser independence condition. If IIA were not satisfied, then these runoff methods would not need to have round-by-round eliminations, since these successive runoffs would not affect the election outcome.

The preferential instant runoff methods also fail the monotonicity condition by nature of the round-by-round eliminations. The design of preferential ballots allow for second, third, and other place votes to affect the outcome of an election, just as the candidate eliminations do. This was illustrated in the Instant Runoff Voting example.

5. The Crowding Problem

As mentioned in section 2.3, the crowding problem refers to the emergence of a two, three, or n-party system that prevents certain candidates from winning an election on the unit interval model regardless of their positions. After discussing the properties of each voting system in the previous section, we will analyze the crowding phenomenon and winning strategies for each method theoretically and empirically.

Ralph Nader's political campaigns are examples of the crowding effect in U.S. Presidential elections. Though he gained popularity in the 2000 and 2004 elections, Nader, an independent unaffiliated with the democrats or republicans, had difficulty persuading people to vote for him in the primaries due to concerns over his "electability." The term "electability" was coined by the media when describing a candidate's chances of winning a head-to-head runoff election for president. The media emphasizes candidates from highly visible and prominent political parties because these parties are perceived as having sufficient political capital to win.

The current two-party system, dominated by democrat and republican candidates, has been in place in the U.S. for over a century. Third-party candidates have faced significant obstacles in drawing voter support away from the dominant parties. Many voters do not want to vote for a candidate they believe has little or no chance of winning. Rather than see their vote go to "waste," they will pass their votes along to one of the dominant candidates, thus creating a rather self-fulfilling prophecy. In the following section, we will analyze the tendency of Plurality and other voting systems to exhibit this crowding phenomenon.

5.1 Single Vote Plurality

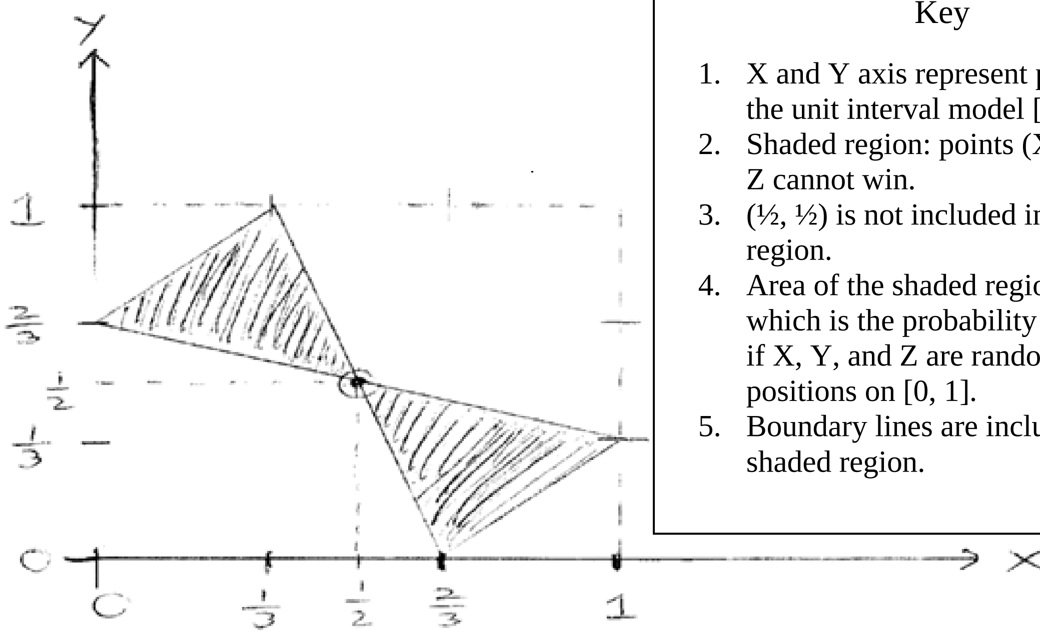
5.1.1 Three Candidates: Strategy and Probability of Crowding Out

To study the crowding problem in the Single Vote Plurality system, we will refer to the unit interval model properties and introduce potential winning strategies for the last candidate to enter an election. In the three-candidate case, we assume the first two candidates, X and Y, have already chosen unique positions on [0, 1]. The last candidate to choose a position, Z, has the following optimal strategies to choose from.

- L: $Z = X – e$ (left of both candidates)
- LC: $Z = X + e$ (left of Y, but right of and next to X)
- R: $Z = Y + e$ (right of Y)
- RC: $Z = Y – e$ (left of and next to Y, right of X)

Through both algebraic and geometric means, assuming X and Y choose their positions on [0, 1] randomly, there is a probability of 2/9 or 22% that Z will not be able to win the election. Winning strategies are determined by choosing positions for Z, given all possible values of X and Y, such that P(Z) > P(X) and P(Z) > P(Y). Figure 1 illustrates these points where X and Y can be such that Z cannot win.

Figure 1: Plurality Graph for Three Candidates



Key

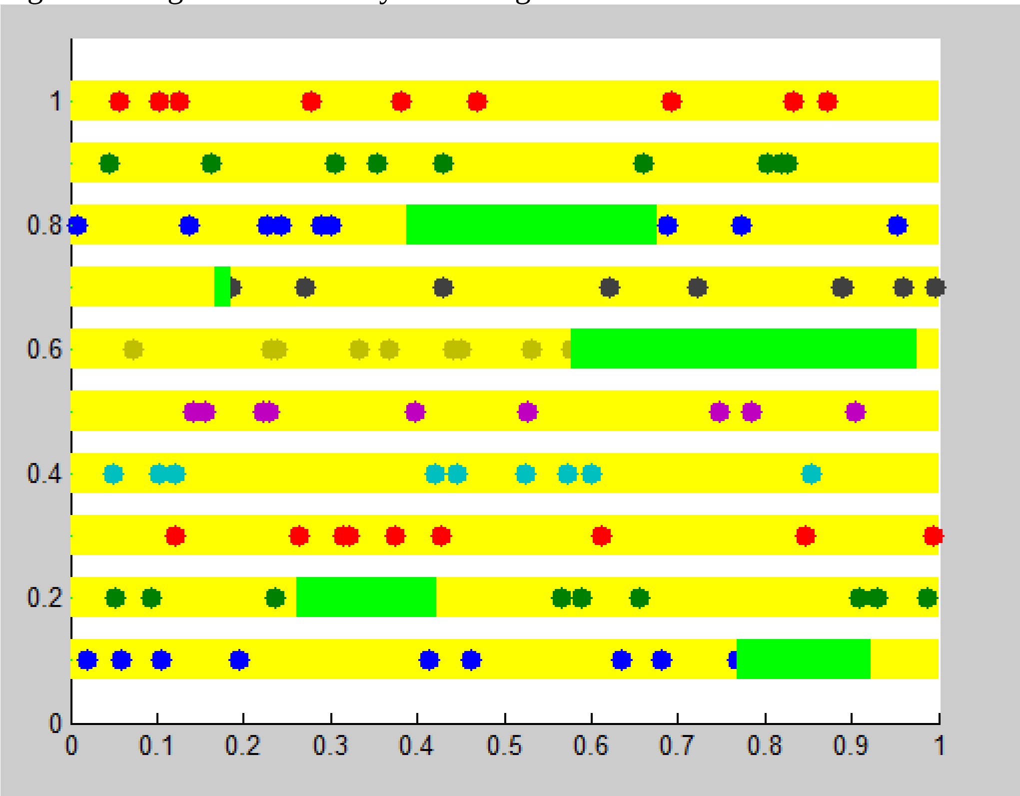
1. X and Y axis represent positions on the unit interval model [0, 1]
2. Shaded region: points (X, Y) where Z cannot win.
3. (½, ½) is not included in the shaded region.
4. Area of the shaded region is 2/9, which is the probability that Z loses if X, Y, and Z are randomly assigned positions on [0, 1].
5. Boundary lines are included in the shaded region.

However, candidates in real elections rarely choose their positions randomly. Thus, plurality creates a significant opportunity for two parties to squeeze out a third, much like the way democrats and republicans have done for decades. The proliferation of the two-party system limits voters' realistic choices, since voting for the independent or third party would be nearly equivalent to throwing away a vote in many cases. This phenomenon causes valuable information about the true preferences of voters to be lost.

5.1.2 Multi-Candidate Strategies

Since it only takes two candidates to effectively crowd out a third, increasing the number of candidates makes it much easier to crowd out the last candidate because fewer votes and possible winning positions are left on the unit interval. To show evidence of further crowding, we simulate random unit interval model elections in MATLAB. The input to the Plurality election function is a vector with the positions of all but one of the candidates. Thus, for an n-candidate election, the length of this vector is n-1. Figure 2 shows the potential winning strategies in a sample of 10 Plurality elections simulated in MATLAB.

Figure 2: Single Vote Plurality Crowding



Each line represents a sample election unit interval model. The multi-colored dots represent the positions the first nine candidates take before the tenth candidate can position himself. The yellow region represents the positions where the last candidate cannot win the election, whereas the green region shows a winning strategy. Though this a very small sample, it is important to note that half of these elections exhibit crowding. Put differently, half of these elections have configurations of candidates which prevent the last candidate from winning, regardless of where this candidate may choose to position himself. However, in order to get a better sense of the frequency of the crowding phenomenon across all election methods for large numbers of candidates, we must use a much greater sample size. Consider the frequency of crowding for a Plurality election with ten total candidates, with a sample size of 1,000 elections. This simulation returns 444 crowding scenarios, implying a 44.4% relative frequency of crowding in Plurality elections with ten total candidates. Since this program was run under the assumption of random positions of candidates, there is an even greater chance of crowding if candidates consciously try to position themselves to prevent the last candidate from winning.

5.2 Instant Runoff Voting

5.2.1 Three Candidates: Strategy and Probability of Crowding Out

In looking at the three-candidate Instant Runoff Voting case, we consider the following eight potentially winning strategies on the unit interval model, which together describe the space of optimal strategies for the third-party candidate (Z).

Strategies (for some small epsilon, $e \approx 0$):

- L: Z = X – e (left of both candidates)
- LC: Z = X + e (left of Y, but right of and next to X)
- R: Z = Y + e (right of Y)
- RC: Z = Y – e (left of and next to Y, right of X)
- M^L_x: Z = 1 – X – e (middle, left of 1-X)
- M^R_x: Z = 1 – X + e (middle, right of 1-X)
- M^L_y: Z = 1 – Y – e (middle, left of 1-Y)
- M^R_y: Z = 1 – Y + e (middle, right of 1-Y)

The IRV case is divided into three sub-cases for Y > X. As alluded to previously, the Y < X case is easily inferred from this set by reflecting the cases about the line Y = X.

Cases for Y > X:

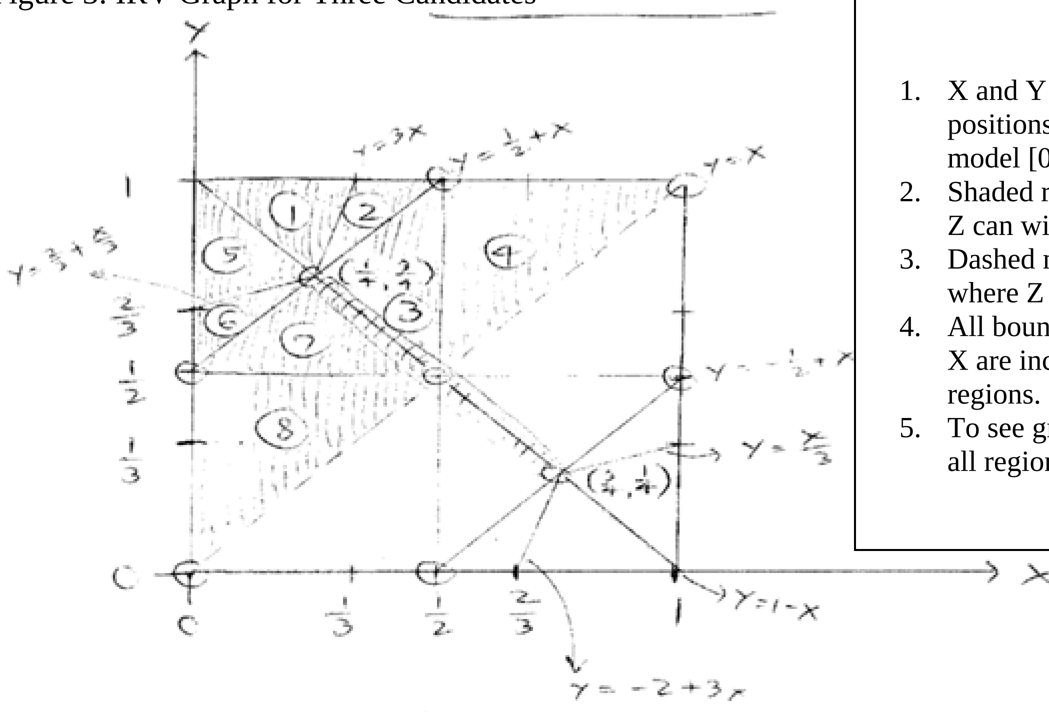
1. 1 – Y < X < Y
2. X < 1 – Y < Y
3. X < Y < 1 – Y

The following graph is similar to the Plurality graph in that it details the possible position combinations for X and Y to try and crowd out Z. However, with the exception of a line and several points, there is no two-dimensional surface area that corresponds to a crowding scenario for Z. Put differently, in the three candidate case, there is generically no way for two candidates to crowd out a third. Winning strategies are listed below and illustrated in figure 3. Note that some combinations of X and Y allow for multiple winning strategies for Z.

Winning Strategies for Regions 1-8 (see Figure 3):

1. LC, M^L_x
2. L, M^L_x
3. L
4. L, M^L_x, M^R_x, M^R_y
5. RC, M^R_y
6. R, M^R_y
7. R
8. R, M^R_y, M^L_y, M^L_x

Figure 3: IRV Graph for Three Candidates



Key

1. X and Y axis represent positions on the unit interval model [0, 1]
2. Shaded regions: points where Z can win.
3. Dashed middle line: points where Z cannot win.
4. All boundaries except for Y = X are included in shaded regions.
5. To see graph for Y < X, reflect all regions about Y = X.

Isolated Cases Where Z Cannot Win:

1. Line segment: **$Y = 1 - X \cap .25 \leq X \leq .75$**
2. Four Points: **(X, Y) = (0, .5) (.5, 0) (1, .5) (.5, 1)**

The union of these two sets is a set of "measure zero."[1] This means that the probability of two candidates randomly falling into a location in this set is virtually zero, as illustrated on the unit square graph. This set's region on the graph has no area, so we can say that, generically, a third-party candidate can always find a way to win in a three-candidate Instant Runoff election. Although it is technically possible for two candidates to position themselves in such a way as described above, the odds of them actually falling exactly on such precise locations are highly unlikely, even if they consciously attempted to do this through collusion. The word "generically" implies that given a configuration where the third-party candidate Z cannot win and some new epsilon ($E > 0$), we can perturb each of the $X_1 \ldots X_n$ by a shift less than E that would provide Z with a way to win.

In addition to finding points where Z cannot win, there exist two special points (X, Y) where Z not only loses the election, but also loses in the first round, regardless of where Z is positioned on [0, 1].

[1] This "set of measure zero" refers to a subset of finite choices, contained within a set of significantly many cases, implying that the probability of randomly landing in one of the cases in this subset is virtually (generically) zero. This is more easily visualized as a subset of several dimensionless points contained in the two-dimensional unit square.

Special Case: **(X, Y) = (.25, .75)**, and by reflection about Y = X, **(X, Y) = (.75, .25)**

- These are the only cases where Z cannot even survive first round elimination.
- P(X) = P(Y) = .5 before Z enters the unit interval.

5.2.2 The General Multi-Candidate Case

Generalizing for $n \geq 3$, we have $\mathbf{(X_1, \ldots, X_n) = (1/2n, \ldots, (2n-1)/2n) = (1 - X_n, \ldots, 1 - X_1)}$, which can be re-written as $\mathbf{X_k = 1 - X_{n-k+1} = (2k-1)/2n}$, where $P(X_1) \ldots P(X_n) = 1/n$. This case gives great insight into the dynamics of elections with virtually unlimited numbers of candidates, leading us to the following propositions.

Theorem 1:

1. $\mathbf{X_k = 1 - X_{n-k+1} = (2k-1)/2n}$ is the only configuration where the last candidate (X_{n+1}) cannot survive first round elimination (proof attached separately).
2. Any small perturbation or set of perturbations (E) will create opening for the last candidate to survive the first round (for E >> e).

We will now prove this conjecture using all previously studies assumptions on the unit interval model. Suppose the first n-1 candidates are distributed $[X_1, \ldots, X_{n-1}]$ such that X_n, the last candidate to enter, is eliminated in the first round. Consider the following strategies for X_n:

A. $X_n = X_1 - e \approx X_1$, as $e \rightarrow 0$

Since, X_n is eliminated first, $P(X_n) < \min [P(X_1), \ldots, P(X_{n-1})] \rightarrow X_1 \leq (X_2 - X_1)/2$

B. $X_n = X_1 + e \approx X_1$, as $e \rightarrow 0$

Since, X_n is eliminated first, $P(X_n) < \min [P(X_1), \ldots, P(X_{n-1})] \rightarrow X_1 \geq (X_2 - X_1)/2$

The only way both of these conditions can be satisfied at the same time is if $X_1 = (X_2 - X_1)/2$. Let $\Delta X_i = (X_i - X_{i-1})/2$. Generalizing these strategies to all the other candidate positions, we have

$X_1 = (X_2 - X_1)/2 = (X_3 - X_2)/2 = \ldots (X_i - X_{i-1})/2 \ldots = (X_n - X_{n-1})/2 = 1 - X_n$
$2\Delta X_1 = \Delta X_2 = \Delta X_3 = \ldots \Delta X_i \ldots = \Delta X_{n-1} = 2\Delta X_n$
So vector $\mathbf{\Delta X} = [½, 1, 1, \ldots, 1, ½]/n$
$\mathbf{\Delta X} = [1/2n, 1/n, 1/n, \ldots, 1/n, 1/2n]$

So the vector of all the candidate positions is
$\mathbf{X} = [X_1, X_2, X_3, \ldots, X_{n-1}] = [1/2n, 3/2n, 5/2n, \ldots, (2n-3)/2n, (2n-1)/2n]$

Generalizing for the k-th candidate, we get
$\mathbf{X_k = 1 - X_{n-k+1} = (2k-1)/2n}$ where $P(X_i) = 1/n \ \forall \ i \in n$.

Thus, any perturbation of this positioning will create an imbalance, resulting in $\mathbf{P_{max}(X_{n+1}) \geq P_{min}(X_{1...}\,X_n)}$. This means candidate X_{n+1} will always find a way to avoid first round elimination if the original configuration $\mathbf{X_k = 1 - X_{n-k+1} = (2k-1)/2n}$ is altered in any way. This case generalizes to each successive round in the runoff election. This is because the second round of an election with n rounds is equivalent to the first round of an election with n-1 rounds of elimination. Each round of the election can be thought of the first round of a smaller election. This implies that if a candidate manages to get past the very first round, they should also be able to get past the successive “first rounds” after that. Thus, the main hurdle for the last candidate to cross is getting past the first round. However, we just showed that this is always generically possible, so theoretically the last candidate should always be able to find a way to win, regardless of the positions and number of the other candidates.

Theorem 2:

After looking at this and other examples, one notices common threads in the best strategies available for the last candidate, regardless of the number and positions of other candidates. The generalized optimal strategy lies in crowding to the left or right of the candidate who would have won the election of the original n candidates (call him/her candidate W). This method of crowding (or moving right next to) W or 1-W does two things: it takes votes away from W and positions the last candidate (Z) to take W's place if and when W is eliminated. There are two possible scenarios:

1. W is eliminated in the first round, so Z takes W's place and thus goes on to win the election.
2. The election rounds proceed as before (as with the first n candidates only) until W is eliminated, then Z replaces W and eventually wins.

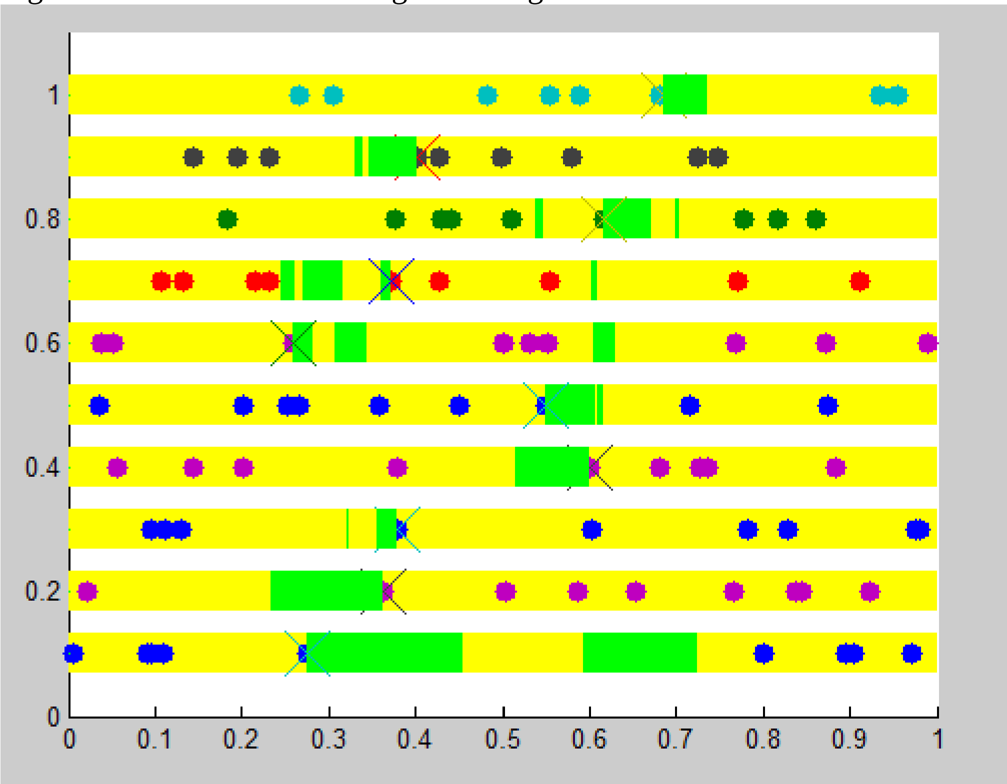
Caveats:

1. Z must make the right choice to move to the left or right of W or 1-W in order to ensure that Z lasts longer than W, by receiving more first-place votes than W. This means that Z must be able to play out the first election of n candidates and make this choice based on which side will result in P(Z) > P(W).
2. The distance between 1/2 and W must be greater than "e" for $Z = W \pm e$ on the unit interval model.
3. If only Z and W are left in the final runoff, Z must be closer to 1/2 than W is.
4. The first election of n candidates must have a clear winner selected (no ties) for Z to compete with.

5.2.3 Multi-Candidate Simulations

To test the theorems discussed in the previous section, we rely on random elections on the unit interval model in MATLAB. Figure 4 shows ten samples of Instant Runoff elections with ten total candidates.

Figure 4: Instant Runoff Voting Crowding



Though ten elections is a small sample, it is striking to see that all elections have regions where the last candidate can position themselves to win, as opposed to only half in the Plurality example. Taking 1,000 samples yields no positions where crowding can occur, which is striking compared to the 444 instances in the Plurality case. Another interesting point lies in the type of winning regions each method propagates. In the Plurality case, all of the winning regions were non-disjoint for each sample election. With Instant Runoff Voting, many sample elections yield disjoint winning regions, demonstrating that there can be multiple winning strategies for each election, regardless of the number and positions of the other candidates. This disjoint attribute results from the iterated nature of the round-by-round eliminations in a runoff election. The X's on each line in the figure represent the would-be winner W among the first nine candidates to position themselves in each election. Each election has a green winning region that is adjacent to W, which is in agreement with the winning strategy Theorem 2 described in the previous section.

5.3 Borda Count

Single Vote Plurality and Instant Runoff Voting seem to give rise to several, consistent winning strategies where winning is possible. It is interesting to study the Borda Count system in terms of common winning strategies in addition to the crowding problem. The Borda margin of victory matrix is assembled by taking head-to-head matchup outcomes from the positions of candidates on the unit interval model. The winner is the candidate whose row in the matrix has the highest sum. Using a sample of ten elections, we get the following Borda graph.

Figure 5: Borda Count Crowding

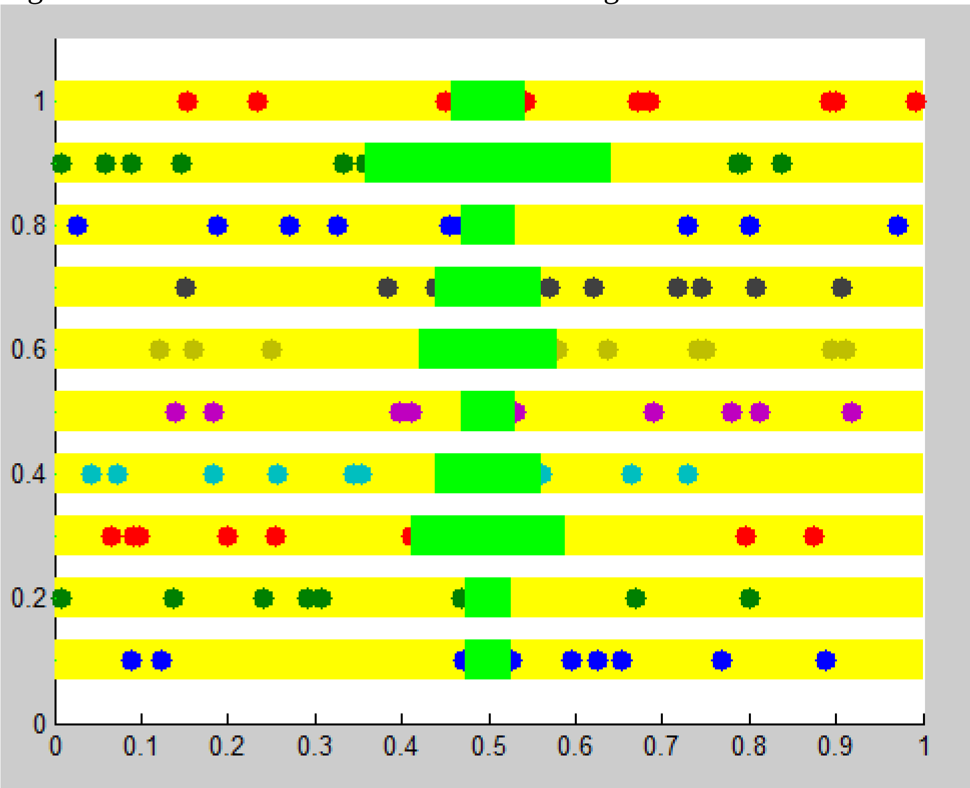
Just as in the case of Instant Runoff Voting, every election in the sample has a winning region for the last candidate in a Borda system. However, the winning strategy does not seem as multi-facet or complex as with the previous two methods. This sample suggests that Borda favors a somewhat centrist winning strategy. Borda does not have an iterated runoff system, so it makes sense that the winning regions in each sample election are non-disjoint. Since Borda is also a margin of victory matrix method, it places more emphasis on head-to-head matchups between candidates, as opposed to Plurality which only considers vote percentages for all candidates at once. In pair-wise matchups between candidates on the unit interval model, the candidate closest to the center wins, so even taking into account all of the other candidates, it is clear that Borda winners generally take positions closer to the center than other candidates. However, this does not mean that merely moving closer to the center will necessarily result in a win.

In a sample of 1,000 elections of ten total candidates, there are no cases where the tenth candidate cannot win. The only way a crowding problem could occur in an election is if one candidate positions himself at exactly 0.5. In practice this is very difficult to accomplish in a competitive election with many candidates.

5.4 Instant Runoff Borda Count

We showed earlier that Instant Runoff Borda satisfies a few properties that the Borda count did not satisfy on its own. Nevertheless, we will see that the additional runoff component does not fundamentally change the winning strategies or increase the prevalence of crowding from the Borda case. The margin of victory matrix is assembled by taking head-to-head matchup outcomes from the positions of candidates on the unit interval model. Each round the candidate whose row (and corresponding column) with the lowest sum is eliminated. The new margin of victory matrix is assembled again from the unit interval model and the winner is the last candidate left. Figure 6 shows ten sample elections for Instant Runoff Borda.

Figure 6: Instant Runoff Borda Count Crowding

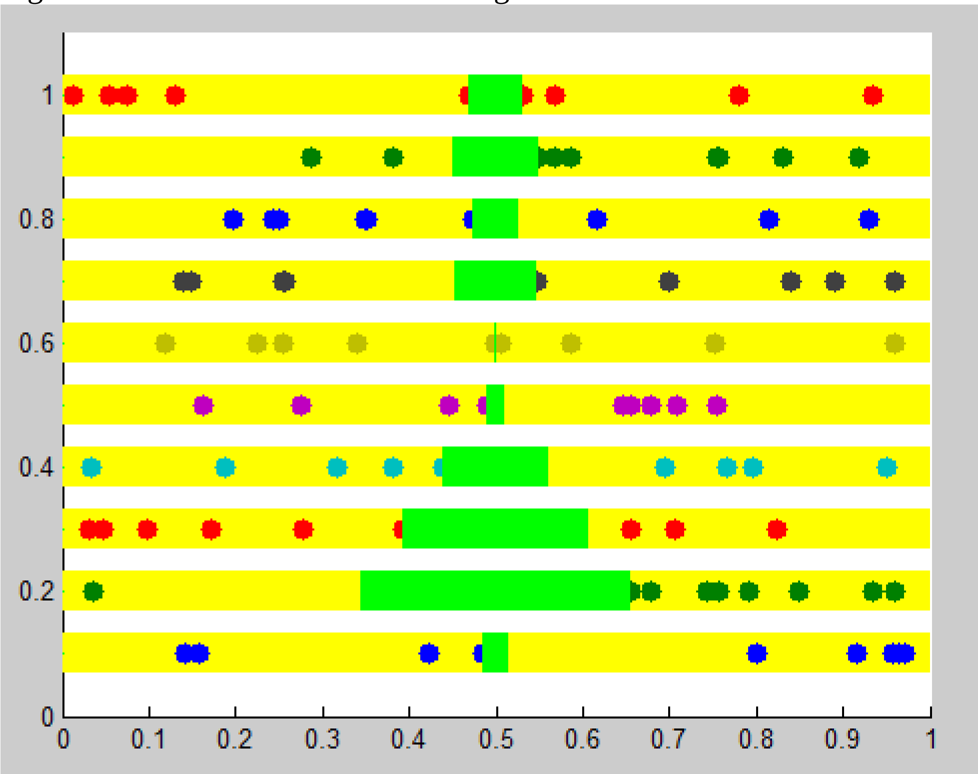


Winning regions for Instant Runoff Borda exhibit purely centrist tendencies. Unlike the case of the Borda Count, a candidate in IRBC can always win by positioning themselves closer to the center than any of the other candidates. This was not necessarily the case in the Borda system. Interestingly, even though IRBC contains a runoff process, the regions are still non-disjoint, unlike the case with Instant Runoff Voting. The regions are non-disjoint most likely because of the centrist component of the winning strategy ingrained in the Borda Count. Even though election outcomes may be different between Borda and Instant Runoff Borda, both methods favor a centrist strategy. Unless one of the candidates is positioned at exactly 0.5, there should always be a way for the last candidate to win. A sample of 1,000 elections confirms this by returning no cases where the last candidate can be crowded out.

5.5 Least Worst Defeat

Like Instant Runoff Borda, Least Worst Defeat also favors a centrist strategy. The margin of victory matrix is assembled from the candidate positions on the unit interval model, just as in the Borda and IRBC systems. The candidates who strive to have the lowest margins of loss will want to position themselves in a way that maximizes their head-to-head performance. On a unit interval model, this implies that they must move close to 0.5 to increase their chances of winning. Figure 7 illustrates this phenomenon below.

Figure 7: Least Worst Defeat Crowding

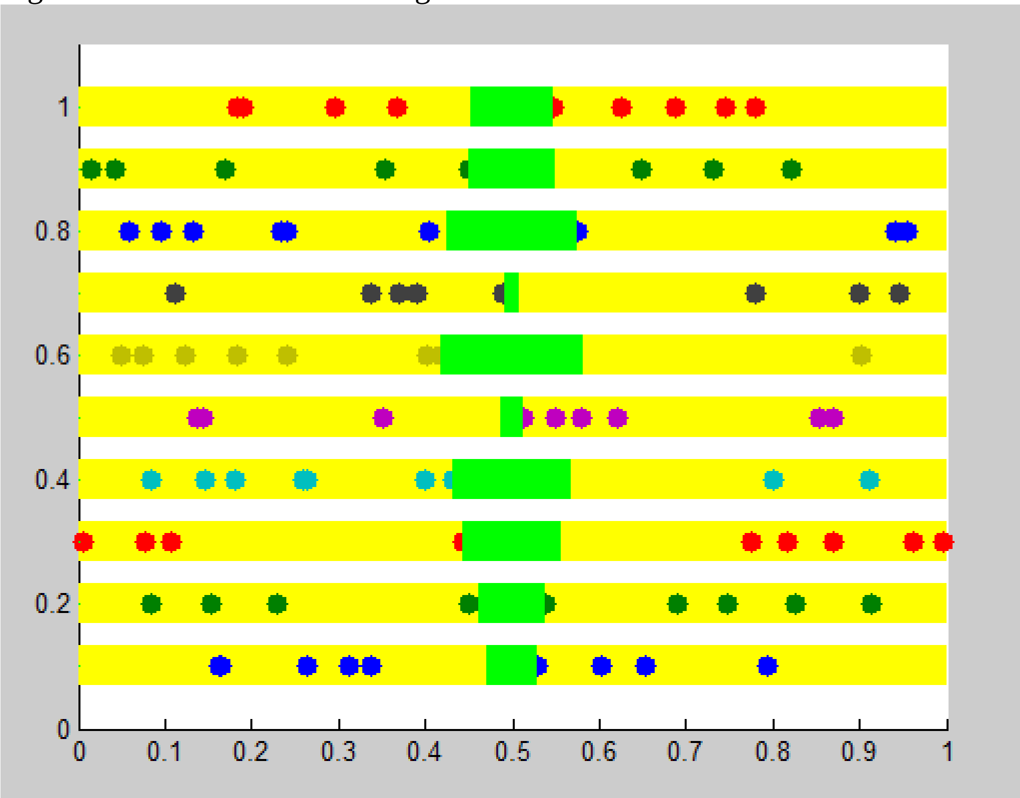


This centrist strategy trend makes sense from the Condorcet winner criterion standpoint, since any candidate that wins all head-to-head matchups on the unit interval model must be closest to the center. We can use this to infer that all Condorcet methods, run using the unit interval model, must incentivize centrist strategies. We can take this one step further and show that all Condorcet methods on the unit interval model do not generically allow for crowding to take place. This is confirmed in our sample of 1,000 elections where no cases of crowding emerge for Least Worst Defeat. In order for centrist strategies to fail, there must be a candidate positioned at exactly 0.5. Excluding this highly unlikely scenario, there should always be an opportunity for the last candidate to enter an election to position himself closer to the center than his opponents. Unlike Instant Runoff Voting, Condorcet methods also provide a more straight-forward winning strategy with non-disjoint winning regions.

5.6 Ranked Pairs

Like the previous Condorcet methods studied, Ranked Pairs favors a centrist winning strategy, which generically allows the last candidate to enter an election to win. The sample of ten elections below is, not surprisingly, similar to the pictures for LWD and IRBC.

Figure 8: Ranked Pairs Crowding



In a sample of 1,000 elections, Ranked Pairs reveals no crowding scenarios, which fits our hypothesis for the lack of crowding in Condorcet methods run on the unit interval model. In the next section we summarize this and other key points gained from this numerical analysis.

5.7 Summary Analysis

The first striking result from the crowding analysis is how the crowding problem becomes magnified for the Plurality method as the number of candidates increases. Even when the last candidate is not crowded out, the potential winning strategies do not follow a coherent, replicable pattern. We contrast this with the Instant Runoff Voting method, showing that, generically, the last candidate can always find a way to win the election. However, the winning strategies are also disjoint and difficult to practically implement, as illustrated by the complicated instructions and caveats in Theorem 2 discussed in sub-section 5.2.2.

The Borda Count method also prevents crowding and incentivizes somewhat centrist winning strategies, though a centrist strategy will not necessarily result in a win. Further improvement is seen when considering the Condorcet methods of Instant Runoff Borda, Least Worst Defeat, and Ranked Pairs. We prove that all Condorcet methods run on the unit interval model produce centrist winning strategies, where positioning closest to the center (0.5) than the other candidates ensures victory. For this reason, provided no other candidate is at exactly 0.5, these Condorcet methods generically prevent crowding. The winning regions are also non-disjoint because they are centrist, making these winning strategies more practical to implement.

Below is a table illustrating the crowding percentage per number of candidates for each voting system. 1,000 sample elections are run for every number of candidates, and the crowding percentage refers to the relative frequency of cases where the nth candidate cannot win. Cases where there is a 0.10% frequency of crowding refer to one election out of 1,000 with a candidate positioned at exactly 0.5. The table illustrates Plurality's propensity for crowding, while the other voting methods are virtually immune to crowding.

Figure 9: Crowding Summary: 1,000 sample elections

Number of Candidates	SVP %	IRV %	Borda %	IRBC %	LWD %	RP %
2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
3	21.40%	0.10%	0.00%	0.00%	0.00%	0.00%
4	26.90%	0.00%	0.00%	0.00%	0.00%	0.00%
5	34.70%	0.00%	0.00%	0.00%	0.00%	0.00%
6	35.20%	0.00%	0.00%	0.00%	0.00%	0.00%
7	42.20%	0.10%	0.00%	0.00%	0.00%	0.00%
8	43.10%	0.00%	0.00%	0.00%	0.00%	0.00%
9	46.00%	0.00%	0.00%	0.00%	0.00%	0.00%
10	47.10%	0.10%	0.10%	0.00%	0.00%	0.00%

6. Top Preferential Ballots: Random Elections

In this section we will compare two of the best and most popular preferential ballot voting methods through simulations. These methods, Least Worst Defeat and Ranked Pairs satisfy more of the most vital conditions than the other methods studied in this paper. Specifically, both of these methods satisfy monotonicity and the Condorcet property. Plurality fails the Condorcet property, as does Instant Runoff Voting, and Borda Count. Instant Runoff Borda, though in compliance with the Condorcet winner criterion, fails the monotonicity condition, along with Instant Runoff Voting. Nevertheless we run simulations for Ranked Pairs and Least Worst Defeat with Borda Count as a reference. Before going into the simulations, we briefly compare LWD and RP together in terms of properties and describe the special cases of three and four-candidate elections.

6.1 Properties: Least Worst Defeat vs. Ranked Pairs

In terms of satisfying the most properties, Least Worst Defeat and Ranked Pairs work better than the previously studied preferential ballot systems. Ranked Pairs satisfies the Condorcet loser criterion, while Least Worst Defeat does not. In an election with many candidates it may not be very likely that a Condorcet loser emerges with margins of loss small enough for them to win a Least Worst Defeat election. However, in theory, this an important condition for a voting method to satisfy in order to ensure the winner of an election represents voters' true preferences. We will also show that in practice Ranked Pairs winners fare far better in head-to-head matchups than Least Worst Defeat winners.

6.1.1 Frequency Derivations

Before we discuss the sample elections, we must use the Condorcet winner condition as a reference frame for the analysis. Since both methods are Condorcet methods, Least Worst Defeat and Ranked Pairs will always agree and return the same winner when a Condorcet winner exists. Thus, when looking for the frequency of disagreement between Least Worst Defeat (LWD) and Ranked Pairs election winners, we consider the set of elections where no Condorcet winner exists. The following derivations use this subset of elections and the simple form of Bayes' rule to arrive at some important properties that hold true for sets of elections regardless of the number of candidates.

Notation:

- $P(A)$ = Probability that LWD and RP winners are the same
- $P(C)$ = Probability that a Condorcet winner exists
- $P(A \cap C)$ = Probability that LWD and RP winners are the same AND a Condorcet winner exists
- $P(A \mid C)$ = Probability that LWD and RP winners are the same GIVEN that a Condorcet winner exists
- $P(C^c) = 1 - P(C)$ = Probability that there is no Condorcet winner

Derivations:

$P(A \cap C) = P(C) * P(A | C) = P(C)$
$= P(A) * P(C | A) \rightarrow P(C | A) = P(C)/P(A)$

$P(C^c | A) = 1 - P(C | A) = 1 - (P(C)/P(A))$

$P(A | C^c) = P(A \cap C^c)/ P(C^c) = (P(A) * P(C^c | A))/(1-P(C)) = (P(A)*(1-(P(C)/P(A))))/(1-P(C)) \rightarrow$
$\rightarrow$ **$P(A | C^c) = (P(A)-P(C))/(1-P(C))$**

Thus, the probability or relative frequency that Least Worst Defeat and Ranked Pairs agree, given that there is no Condorcet winner, is equal to the difference between the probability that they agree and the probability of a Condorcet winner, divided by the probability that no Condorcet winner exists. This result will provided added insight when studying the simulated data in the multi-candidate case for up to 20 candidates.

6.2 Three Candidate Case: Least Worst Defeat vs. Ranked Pairs

The three candidate election is a special case that reveals an interesting relationship between LWD and Ranked Pairs. Since both systems satisfy the Condorcet winner criterion, we prove that they will always return the same winner in the three candidate case. If an election has a Condorcet winner, it is clear that both methods will agree on the winner. Thus, we consider two situations where no Condorcet winner exists. The first case involves a margin of victory matrix where two of the rows have elements that sum to a positive number and the second case has two rows that sum to a negative number. By its construction, a "three-by-three" margin of victory matrix cannot have all three rows sum to a positive number or all three rows sum to a negative number. Thus, these two cases can be generalized for any arbitrary ordering of candidates and margins of victory. Consider the following margin of victory matrix with the margins of victory satisfying A > B > C.

Case 1 MOV:	0	A	-B	(sum = A - B > 0)
	-A	0	C	(sum = C - A < 0)
	B	-C	0	(sum = B - C > 0)

Matchup Outcome	Margin of Victory
1 > 2	A
3 > 1	B
2 > 3	C

*Final Ranked Pairs ranking: 3 > 1 > 2

Since the lowest margin of victory is C, the LWD winner is candidate three (row 3). Similarly, the rankings of the margins of victory result in candidate three to be elected as the Ranked Pairs winner.

Case 2 MOV:				
	0	A	-C	(sum = A - C > 0)
	-A	0	B	(sum = B - A < 0)
	C	-B	0	(sum = C - B < 0)

Matchup Outcome	**Margin of Victory**
1 > 2	A
2 > 3	B
3 > 1	C

*Final Ranked Pairs ranking: 1 > 2 > 3

The Ranked Pairs winner is candidate one and the LWD winner is also candidate one by virtue of having the lowest defeat margin of C. Rearranging the candidates or positions of the margins of victory in the matrix for the ranking A > B > C will not change the outcome of the elections. The Ranked Pairs winning row will always contain the lowest, worst defeat of all the candidates. Since the ranking of margins of victory is arbitrary, we conclude that Ranked Pairs and LWD always agree on a winner in the three candidate case.

6.3 Multi-Candidate Random Elections

6.3.1 Least Worst Defeat vs. Borda Count

To get a sense of the performance of Condorcet methods relative to non-Condorcet methods, we use random elections and count the frequency of agreements and disagreements between the winners from each election. If the winners are different between two of these election methods, then we check to see which of the two winning candidates beats the other head-to-head. The voting system that produced the winner of this head-to-head matchup is viewed as the stronger method in that case, since it more accurately takes in and expresses voter preferences for candidates. Running thousands of simulations such as these can yield a trend that suggests one method can be significantly better than another.

Consider the following Condorcet table and agreement table between Least Worst Defeat and Borda Count, assembled using 10,000 random elections of 1,000 voters each.

Figure 10: Frequency of Condorcet winner

Number of Candidates	Condorcet %
3	90.57%
4	82.80%
5	75.10%
6	68.01%
7	62.90%
8	58.28%
9	54.33%
10	51.37%
11	48.40%
12	46.05%
13	43.79%
14	40.95%
15	39.28%
16	37.08%
17	36.07%
18	34.39%
19	33.50%
20	31.85%

Figure 11: Frequency of Agreement and Wins between LWD and Borda

Number of Candidates	Agreement %	LWD Win %	Borda Win %
3	88.56%	100.00%	0.00%
4	81.66%	89.20%	10.80%
5	78.74%	84.95%	15.05%
6	75.35%	81.74%	18.26%
7	73.59%	77.36%	22.64%
8	71.89%	75.45%	24.55%
9	69.65%	73.25%	26.75%
10	68.45%	72.04%	27.96%

For the three candidate case, Borda and LWD seem to agree almost 90% of the time. However, this percentage decays significantly as the number of candidates increases. This makes sense, since we would not expect a Condorcet method to always agree with a non-Condorcet method for even a small number of candidates. More importantly, we want to test whether LWD emerges as the more dominant method of these two, as suggested by our analysis of the properties. To analyze this, we compare the winning percentages of one against the other for each given number of candidates.

For three to ten candidates, our sample shows that the LWD winner beats the Borda winner in a head-to-head matchup in over 72% of cases where the two methods disagree. For the three candidate case, LWD will always beat the Borda winner if there is a disagreement. Since there is a Condorcet winner about 90% of the time, that winner will beat the Borda winner if the Borda winner is not a Condorcet winner, or tie the Borda winner if the Borda winner is a Condorcet winner. For the remaining 10% of elections without a Condorcet winner, we will prove that the LWD will also either beat or tie the Borda winner. Consider the following election margin of victory matrix with A > B > C as the margins of victory. This election does not have a Condorcet winner.

Case 1 MOV:	0	A	-B	(sum = A - B > 0)
	-A	0	C	(sum = C - A < 0)
	B	-C	0	(sum = B - C > 0)

Matchup Outcome	**Margin of Victory**
1 > 2	A
3 > 1	B
2 > 3	C

For this three candidate case, candidate three is the LWD winner because it has the lowest margin of loss. The Borda winner is the candidate that corresponds to the row with the highest sum of their margins of victory. Since the sum of row two is negative, the only potential Borda winners are candidates one and three. Which one of these is the winner depends on how much larger A is than B, relative to how much larger B is than C. If A – B > B – C, then candidate one is the Borda winner. However, this candidate loses head-to-head to candidate three, the LWD winner. If A – B < B – C, then candidate three is the Borda winner and the two voting systems agree.

Case 2 MOV:	0	A	-C	(sum = A - C > 0)
	-A	0	B	(sum = B - A < 0)
	C	-B	0	(sum = C - B < 0)

Matchup Outcome	**Margin of Victory**
1 > 2	A
2 > 3	B
3 > 1	C

*Final Ranked Pairs ranking: 1 > 2 > 3

Candidate one is the LWD winner in the second case. Since row one is the only row with a positive sum, Borda also elects candidate one as the winner. As in the cases discussed in section 6.4.1, the positions of margins of victory within the matrix are arbitrary. Therefore, regardless of the presence of a Condorcet winner, the LWD winner will always beat the Borda winner in an election with three candidates when the two methods do not agree on a winner.

Note that the LWD win percentage decreases as the number of candidates increases. We expect this trend to occur because the number of Condorcet winners decreases as more candidates participate, making it less likely that LWD will select the dominant Condorcet winner that would beat or tie a Borda winner.

6.3.2 Ranked Pairs vs. Borda Count

Since Ranked Pairs is another Condorcet method, we expect it to exhibit a similarly strong winning percentage against the non-Condorcet Borda Count system. We perform the same simulation of 10,000 elections and 1,000 voters per election to analyze this relationship.

Figure 12: Frequency of Agreement and Wins between Ranked Pairs and Borda

Number of Candidates	Agreement %	RP Win %	Borda Win %
3	88.56%	100.00%	0.00%
4	81.49%	92.76%	7.24%
5	78.15%	90.94%	9.06%
6	74.87%	89.38%	10.62%
7	72.13%	86.40%	13.60%
8	70.75%	85.95%	14.05%
9	68.19%	84.19%	15.81%
10	66.84%	85.07%	14.93%

Ranked Pairs and Borda share very similar agreement levels for each number of candidates to the levels exhibited between LWD and Borda. Also like LWD, Ranked Pairs winners beat Borda winners a significant percentage of the time. However, this win percentage tends to plateau from seven to ten candidates and is higher for each number of candidates greater than three.

In the three-candidate case, Ranked Pairs winners beat the Borda winners every time the methods return different winners. If a Condorcet winner exists and the methods disagree, then the Ranked Pairs Condorcet winner will clearly beat the non-Condorcet Borda winner. We proved in section 6.2 that LWD and Ranked Pairs will always agree for elections with three candidates. Since Ranked Pairs and LWD always agree regardless of whether or not a Condorcet winner exists, we conclude that Ranked Pairs winners will always beat Borda winners when the methods disagree in the three candidate case.

6.3.3 Least Worst Defeat vs. Ranked Pairs

Ranked Pairs and LWD always agree for the three candidate case, but what does this relationship look like for more candidates? Which method wins more head-to-head matchups when they disagree? To answer these questions, we run 10,000 sample elections with 1,000 voters each to arrive at the frequencies in the table below (figure 13). The table in figure 14 and graph in figure 15 show the Condorcet existence, agreement, and winning percentages for up to 20 candidates and 10,000 voters per each election.

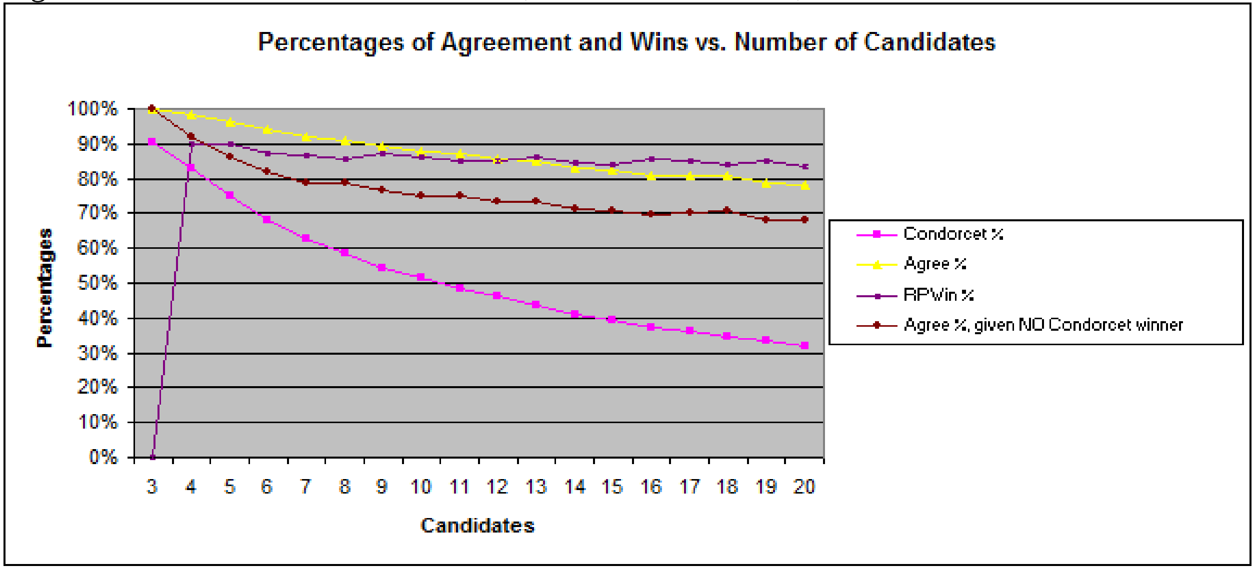
Figure 13: Ranked Pairs vs. LWD for 10,000 elections and 1,000 voters

Number of Candidates	Agreement %	RP Win %	LWD Win %
3	100.00%	N/A	N/A
4	98.50%	90.00%	10.00%
5	96.19%	86.61%	13.39%
6	94.65%	83.55%	16.45%
7	92.30%	86.23%	13.77%
8	91.21%	87.71%	12.29%
9	89.19%	85.29%	14.71%
10	87.72%	87.70%	12.30%

Figure 14: Percentages for Random Samples with 10,000 Elections and 10,000 voters

Number of Candidates	RPwin %	Agree%	Condorcet%	% Agree given NO Condorcet
3	N/A	100.00%	90.57%	100.00%
4	90.00%	98.60%	82.80%	91.86%
5	89.88%	96.54%	75.10%	86.10%
6	87.33%	94.16%	68.01%	81.74%
7	86.93%	92.12%	62.90%	78.76%
8	85.76%	91.22%	58.28%	78.95%
9	87.19%	89.23%	54.33%	76.42%
10	86.33%	87.93%	51.37%	75.18%
11	85.29%	87.08%	48.40%	74.96%
12	84.98%	85.62%	46.05%	73.35%
13	85.98%	85.09%	43.79%	73.47%
14	84.54%	83.05%	40.95%	71.30%
15	83.99%	82.32%	39.28%	70.88%
16	85.53%	80.99%	37.08%	69.79%
17	85.18%	80.90%	36.07%	70.12%
18	84.03%	80.90%	34.39%	70.89%
19	85.04%	78.75%	33.50%	68.05%
20	83.64%	78.24%	31.85%	68.07%

Figure 15: Ranked Pairs vs. LWD for 10,000 elections and 10,000 voters



The agreement percentages between LWD and Ranked Pairs are much higher than the agreement frequencies between LWD and Borda and the agreement percentages between Ranked Pairs and Borda. It is striking how slowly the overall agreement and conditional agreement (conditioned on no Condorcet winner existing) percentages decay as the number of candidates increases. The Condorcet winner percentage decreases rapidly, while the agreement percentages stay much higher as the number of candidates increases. In the three candidate case, the overall agreement and Condorcet percentages are 100% and about 91%, respectively. For 20 candidates, these numbers are 78% and 32% respectively, and the conditional agreement percentage still remains high at 68%. Thus, regardless of whether or not a Condorcet winner exists, Ranked Pairs and LWD agree on a winner a significant percentage of the time. Since most major elections are run with fewer than ten candidates for practicality, these results are even more telling, as the overall agreement percentage is at least 88% and the agreement percentage given no Condorcet winner is at least 75%.

Another interesting trend to consider is the Ranked Pairs winner's head-to-head winning percentage against the LWD winner. For 20 candidates and under, Ranked Pairs winners beat LWD winners in pair-wise matchups at least 83% of the time. This percentage starts at 90% for four candidates, decreases somewhat as the number of candidates participating increases until about 11 candidates, and then reaches a plateau. Though both methods satisfy the Condorcet criterion and other common properties, Ranked Pairs emerges as the more dominant method that better expresses voter preferences by virtue of winning more head-to-head matchups against LWD winners. It may be interesting to consider a hybrid method between the two elections, which always elects the head-to-head winner between the LWD and Ranked Pairs winners, since Ranked pairs winners do not always beat LWD winners. We will mention this briefly in section 7.2, "Suggestions for Further Research."

Nevertheless, both methods have a high agreement percentage for ten candidates and under. We can infer that, using random elections with margin of victory matrices, Condorcet methods will agree a high percentage of the time for this reasonable number of candidates. Depending on other factors affecting the voter population, it may not even matter which method is used since they agree so often. In fact, Ranked Pairs may not be practical to implement if voters do not properly understand how the system works. Since the method involves programming an interpretation of the transitive property, it is quite complex in terms of coding and run time and thus may not be the most feasible method to understand and implement on a large scale with hundreds of thousands or millions of voters. It is clear, however, that both of these Condorcet methods are among the most effective and representative voting systems and should be heavily considered for any election.

7. Conclusion

7.1 Closing Recommendations

Choosing a practical, understandable, and feasible election method is no easy task. In this paper we studied six voting systems in terms of the properties they satisfy and their susceptibility to the crowding phenomenon. We then narrowed the focus to two Condorcet methods that satisfied monotonicity. In short, there is no easy answer to the question, "what is the best voting method?" There many different ways of looking at how effective voting systems are at representing voter preferences, whether through lack of crowding, head-to-head wins between election winners, or other metrics. Different voter populations may place different emphasis on various properties and conditions that methods need to satisfy. Thus, the methods and properties we valued most highly in this paper may not work for all groups.

Nevertheless, we make the recommendation that, having satisfied the most important properties, generically avoided the crowding problem, and having the highest head-to-head win percentage when election winners disagreed, Ranked Pairs should be the method of choice. This method performed among the best in all phases of our analysis, and should thus be a major consideration for any election.

However, Least Worst Defeat is not far behind in terms of its fundamental properties and lack of crowding issues. Least Worst Defeat is also a much simpler system to program and implement, so it should have significant value for large voter populations. The run time for Ranked Pairs increases dramatically with the number of candidates and voters, while Least Worst Defeat's run time increases much more gradually. Thus, Least Worst Defeat tends to be the more feasible method for elections with many candidates and voters, while Ranked Pairs runs more efficiently for fewer candidates and voters.

Though we typically consider small numbers of candidates for major political elections, it is important to consider more abstract scenarios where candidates may not necessarily be people, but rather policies, bills, or choices in general. When governing bodies are faced with a multitude of choices, the method of choosing which options are best must be practical to implement given a potentially short time interval. For this reason, Least Worst Defeat should always be a top consideration, though Ranked Pairs would be the better method for a smaller number of voters and candidates.

Because the field of voting theory is still being developed, there are many questions left unanswered and other election methods to consider. For this paper, it was necessary to narrow the scope and perform a thorough analysis on a smaller group of methods. The next section describes ideas for further research that should yield more information about election methods.

7.2 Suggestions for Further Research

Though this research paper is meant to provide additional insight into how to qualitatively and quantitatively analyze voting systems, we look to expand its scope and gain further knowledge about comparing these systems.

Since we mentioned briefly that different circumstances and groups of voters may have different preferences for certain election properties, it would be interesting to see under what conditions certain voting methods would perform most effectively. These conditions can be broken down by number of candidates, presence of a Condorcet winner, presence of a Condorcet loser, presence or lack of a simple majority, number of voters, etc. Breaking down what election methods are best in each scenario would be more pragmatic in making a policy recommendation, whether at the student government, county, state, or national level.

To further expand our comparisons between voting systems, it is important to consider more academic and complex, but prominent systems, such as Instant Runoff Least Worst Defeat, Kemeny-Young, and Schulze. Bringing them into our analysis of agreement and head-to-head matchups between winners would grant further insight about what methods yield the most dominant winners.

It would also be useful to research these methods in the context of other properties, such as clone invariance. Clone invariance implies that the rankings of other candidates in an election are not affected by duplicates of one candidate. Given this definition, clone invariance did not fit well into our discussion of crowding, since we assumed at the start that all candidate positions chosen must be unique. However, this may not be a realistic assumption for state or local elections with many candidates, so further research should be done to see which methods satisfy this property and see in what scenarios this property is most important.

Finally, it would be interesting to adjust the analysis performed in this paper according to a more realistic model for voter behavior and distribution. A uniform distribution of voters on the unit interval is not necessarily the most realistic model, so it would be valuable to consider other possible voter distributions, such as the Gaussian. Additionally, purely random elections are virtually non-existent in modern politics, so adapting the random election model to account for implicit voting structures and parties may be beneficial.

8. References

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9. Appendix

Figure 1: Counts and Percentages for Random Sample with 1,000 Voters and 10,000 Elections

Candidates	Condorcet	Agree	Disagree	RP Win	LWD Win	RPwin %
3	9139	10000	0	0	0	N/A
4	8209	9836	164	141	23	85.98%
5	7565	9637	363	335	28	92.29%
6	6825	9451	549	494	55	89.98%
7	6205	9232	768	656	112	85.42%
8	5847	9067	933	804	129	86.17%
9	5474	8980	1020	867	153	85.00%
10	5075	8754	1246	1072	174	86.04%
11	4790	8650	1350	1167	183	86.44%
12	4563	8599	1401	1173	228	83.73%
13	4287	8373	1627	1368	259	84.08%
14	4128	8310	1690	1456	234	86.15%
15	3884	8195	1805	1513	292	83.82%
16	3752	8162	1838	1549	289	84.28%
17	3590	8011	1989	1691	298	85.02%
18	3525	8022	1978	1656	322	83.72%
19	3300	7902	2098	1759	339	83.84%
20	3263	7881	2119	1756	363	82.87%

Figure 2: Graph of Counts for Random Sample with 1,000 Voters and 10,000 Elections