Name:

Final Exam

Complex Analysis, Math 333

December 12, 2013

Show all of your work to get full credit on each problem. You may write on the backs of pages if you need extra space. Good luck!

Problem 1. State and prove the Cauchy-Goursat theorem.

Problem 2. Prove the fundamental theorem of algebra that a degree n polynomial has exactly n zeroes counting multiplicities. (You may not assume the fact that every nonconstant polynomial has at least one zero, but you may assume Liouville's theorem or Rouche's theorem.)

Problem 3. (Generalization of Liouville's Theorem) Suppose that $f(z)$ is entire (analytic in the complex plane) and that

$$|f(z)| \leq A|z|^{7/3} + B$$

for all z and for some real-valued positive constants A and B. Prove that $f(z) = az^2 + bz + c$ for some a, b, c.

(Hint: Use the estimate on the circle of radius R centered at z_0 that $|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}$ when $f(z)$ is analytic inside the circle and M_R is the maximum value of $|f(z)|$ on the circle.)

(b) Why isn't $f(z) = z^{3/2}$ a counterexample to the above theorem? Explain.

Problem 4. Find the Laurent series for

$$f(z) = \frac{z}{(z-i)^2}$$

around $z = i$.

Problem 5. Let $f(z)$ be an analytic function defined on the entire complex plane. State and prove Taylor's theorem for $z_0 = 0$, and prove that the radius of convergence of the power series is infinite.

Problem 6. Let C be the circle of radius 1 centered at $z_0 = 3$. Compute

$$\int_C \cot z \, dz.$$

Problem 7. Suppose $f(z)$ and $g(z)$ are both analytic in a neighborhood of z_0. Suppose $f(z)$ has an order two zero at $z = z_0$ and $g(z)$ has an order three zero at $z = z_0$. Compute a general formula for the residue of the pole of $h(z) = f(z)/g(z)$ at $z = z_0$ in terms of f, g, and their derivatives, evaluated at z_0.

Problem 8. Using residues, compute
$$\int_0^\infty \frac{dx}{(x^2+1)^3}.$$

Problem 9. Let $f(z) = z^6 + 15z^5 + 150z^3 + 200z + 1$.

(a) How many zeros of $f(z)$ are inside the circle of radius 1?

(b) How many zeros of $f(z)$ are inside the circle of radius 2?

(c) How many zeros of $f(z)$ are inside the circle of radius 4?

(d) How many zeros of $f(z)$ are there in the entire complex plane?

Problem 10. Find the residues at $z = 0$ of each of the functions below, and say if this singularity is a removable singularity, an essential singularity, or a pole. If it is a pole, then say what the order of the pole at $z = 0$ is.

(a) $\frac{1}{\sin z}$

(b) $\frac{e^z}{z^3}$

(c) $z^2 \sin(1/z)$

Problem 11. Find the residues at $z = 0$ of each of the functions below, and say if this singularity is a removable singularity, an essential singularity, or a pole. If it is a pole, then say what the order of the pole at $z = 0$ is.

(a) $\frac{\sin(z)}{z^5+z}$

(b) $\frac{\sin(z)}{z^5}$

(c) $\frac{e^z}{\cos z}$, for $z \neq 0$

Problem 12. Compute the value of

$$\int_0^\infty \frac{\sin x}{x} dx$$

by using an indented contour integral around the boundary of a half disk.