Problem	Points	Score
(name)	20	
1	10	
2	5	
3	5	
4	10	
5	10	
6	5	
7	10	
8	15	
9	15	
10	15	
11	15	
12	15	
Total	150	

Name (20 points):

Final Exam

Complex Analysis, Math 333

December 12, 2014

There are 150 points (including 20 points for your name - be careful!) on this 180 minute exam. Show all of your work to get full credit on each problem. You may write on the backs of pages if you need extra space. Good luck!

1. (10 points) Evaluate

$$\int_C \exp(\sin(z^2)\cos(z))dz$$

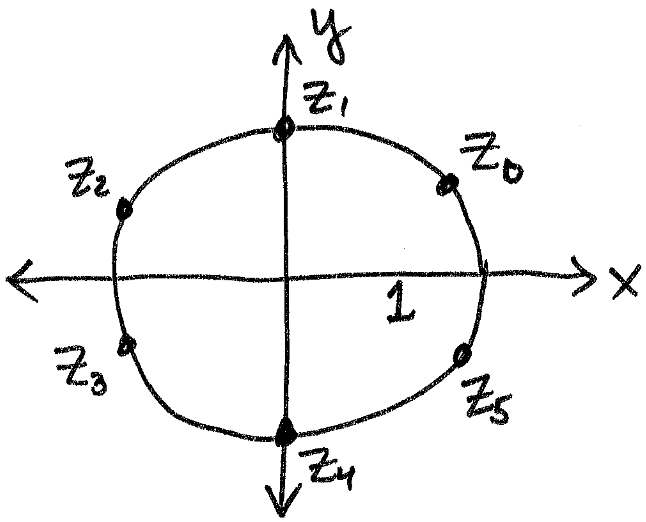
where C is the circle of radius 10 centered at $14+5i$ by any means of your choosing. Justify your answer.

Since $f(z) = e^{\sin(z^2)\cos z}$ is analytic,

$\int_C f(z)dz = \boxed{0}$ by Cauchy-Goursat

since C is the boundary of a region.

2. (5 points) Plot the sixth roots of -1 and express each of the roots in polar coordinates.

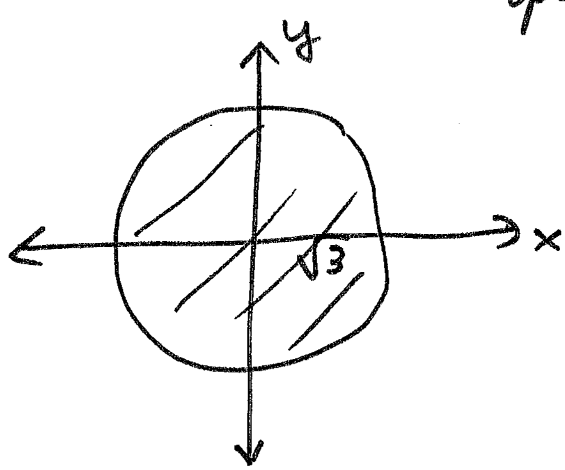


$$z_k = e^{i\left(\frac{\pi}{6}+\frac{\pi}{3}k\right)}, \quad k=0,\dots,5.$$

3. (5 points) Fully describe and graph the region determined by $z\bar{z} < 3$.

$z\bar{z} = |z|^2$, so $|z|^2 < 3 \longrightarrow |z| < \sqrt{3}$

open disk of radius $\sqrt{3}$



4. (10 points) Let $f(z) = \cot(2z)$. $= \dfrac{\cos(2z)}{\sin(2z)} = \dfrac{\cos(2z)}{2z - \frac{1}{3!}(2z)^3 + \frac{1}{5!}(2z)^5 - \ldots}$

(a) What is the order of the pole of $f(z)$ at $z = 0$?

Pole of order $\boxed{1}$ $\leftarrow$ $= \dfrac{\left(\frac{\cos(2z)}{2 - \ldots}\right)}{z} = \dfrac{\emptyset(z)}{z}$

(b) What is the residue of this pole?

where $\emptyset(0) = \frac{1}{2}$.

$\operatorname*{Res}_{z=0} f(z) = \boxed{1/2}$

5. (10 points) Suppose that $f(z)$ is an analytic function on the entire complex plane. Suppose also that $f(-3+i) = 1 + 2i$. Prove that all of the zeroes of $f(z)$ are isolated zeroes (meaning that the function is nonzero everywhere in some neighborhood of the point where $f(z)$ equals zero). Show all details and where all of the hypotheses get used in the proof.

Let z_0 be a zero of $f(z)$ $(f(z_0) = 0)$. Then ~~Exp~~ since $f(z)$ is entire,

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots \quad \text{for all } z$$

$$= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + a_{n+2}(z - z_0)^{n+2} + \ldots$$

for some $n \geq 1$, where $a_n \neq 0$, since $a_0 = f(z_0) = 0$ and not all of the $\{a_k\}$ are zero since f is not identically zero.

$$\therefore f(z) = (z - z_0)^n (a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \ldots)$$

$\uparrow$ only zero at z_0 $\uparrow$ analytic and $\neq 0$ around z_0 $\longrightarrow$ $\boxed{\text{isolated zero}}$

6. (5 points) Compute *all* of the values of $\exp(3\log(2i)/2)$ and simplify as much as possible. How many distinct values are there?

$$\log(2i) = \ln 2 + i\left(\tfrac{\pi}{2} + 2\pi n\right)$$

$$\tfrac{3}{2}\log(2i) = \tfrac{3}{2}\ln 2 + i\left(\tfrac{3\pi}{4} + 3\pi n\right)$$

$$e^{\frac{3}{2}\log(2i)} = 2^{3/2} e^{i\left(\frac{3\pi}{4} + 3\pi n\right)} = \pm 2^{3/2} e^{i\frac{3\pi}{4}} = \pm 2\sqrt{2}\left(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} i\right)$$

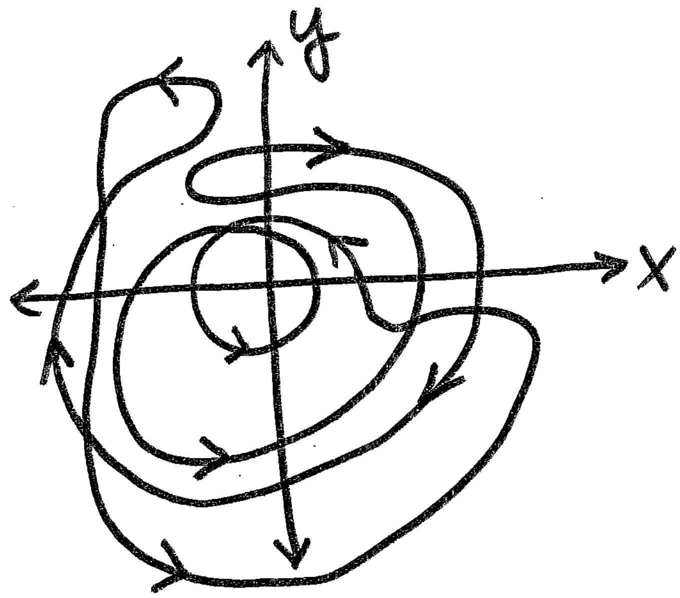
$$= \boxed{\pm 2(-1+i)} \longrightarrow 2 \text{ values}$$

7. (10 points) Let C be the circle or radius 5 centered at the origin and going counterclockwise. Suppose that $f(z)$ is entire and that the image of C under the mapping f is the curve drawn below. Counting multiplicity, how many zeroes does $f(z)$ have inside C and why?

By the Argument Principle,

zeros of f inside C = winding # of f(C) around the origin

= 2



f(C)

8. (15 points) Let $f(z) = \frac{1}{z-1} + \frac{1}{z+1}$.

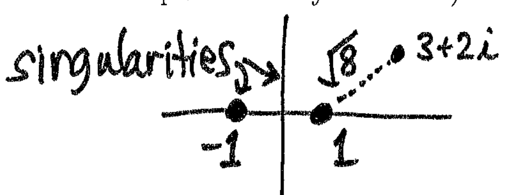
(a) Using the standard formula $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$, for $|x| < 1$, compute the Taylor series for $f(z)$ around $z_0 = 0$. What is the radius of convergence for this series?

$$\frac{1}{z-1} = \frac{-1}{1-z} = -1 - z - z^2 - z^3 - \ldots \quad |z| < 1$$

$$\frac{1}{z+1} = \frac{1}{1-(-z)} = 1 - z + z^2 - z^3 + \ldots \quad \text{Hence,}$$

$$\boxed{f(z) = -2z - 2z^3 - 2z^5 - \ldots}$$ is the Taylor series which has radius of convergence equal to $\boxed{1}$.

(b) What is the radius of convergence of the Taylor series of $f(z)$ around $z_0 = 3 + 2i$? (Do not compute the Taylor series.)



Since $f(z)$ is analytic in $B_{\sqrt{8}}(3+2i)$, the radius of convergence is $\sqrt{8} = 2\sqrt{2}$.

(c) Compute the Laurent series for $f(z)$ in the region $|z| > 1$. $\longleftrightarrow$ $\left|\frac{1}{z}\right| < 1$ (key point)

$$f(z) = \frac{1/z}{1 - 1/z} + \frac{1/z}{1 + \frac{1}{z}}$$

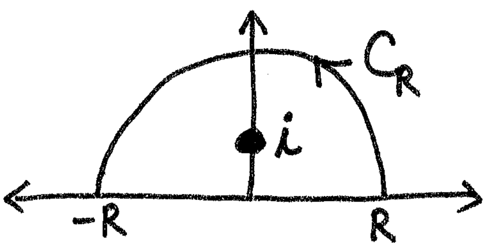
$$= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right) + \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \ldots\right), \quad \left|\frac{1}{z}\right| < 1$$

$$= \boxed{\frac{2}{z} + \frac{2}{z^3} + \frac{2}{z^5} + \ldots}, \text{ for } |z| > 1.$$

9. (15 points) Use a contour integral to evaluate

$$\int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx. = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$$

Include all details for full credit.



Let $f(z) = \dfrac{z^2}{(z^2+1)^2}$ which has singularities at $z = \pm i$. By the Cauchy Residue Theorem,

$$\lim_{R\to\infty} \left(\int_{C_R} f(z)\,dz + \int_{-R}^{R} \frac{x^2}{(x^2+1)^2}\,dx \right) = 2\pi i \operatorname*{Res}_{z=i} f(z)$$

① $$f(z) = \frac{z^2}{(z+i)^2(z-i)^2} = \frac{\phi(z)}{(z-i)^2} \text{ where } \phi(z) = \frac{z^2}{(z+i)^2}. \text{ Hence,}$$

$$\operatorname*{Res}_{z=i} f(z) = \phi'(i) = \left.\frac{2z}{(z+i)^2} - 2\frac{z^2}{(z+i)^3}\right|_{z=i} = \frac{2i}{(2i)^2} - 2\frac{(-1)}{(2i)^3} = -\frac{i}{2} + \frac{i}{4} = -\frac{i}{4}$$

② $$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} |f(z)|\,|dz| = \int_{C_R} \frac{|z|^2}{|z^2+1|^2}\,|dz| \le \int_{C_R} \frac{R^2}{(R^2-1)^2}\,|dz|$$

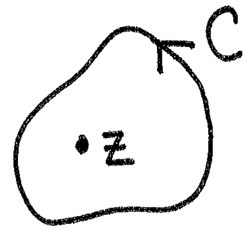
$$= \frac{R^2}{(R^2-1)^2}\cdot 2\pi R \searrow 0 \text{ as } R\to\infty.$$

③ Thus,

$$\int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = \frac{1}{2}\cdot 2\pi i\cdot\left(-\frac{i}{4}\right) = \frac{\pi}{4}.$$

10. (15 points)

(a) State the Cauchy Integral Formula.



For analytic functions f,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)\,dw}{w-z}$$

where C is any simple closed contour going around z once in the clockwise direction.

(b) State the derivative version of the Cauchy Integral Formula.

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)\,dw}{(w-z)^{n+1}}$$

(c) Prove Liouville's theorem (that a bounded entire function must be constant).

Suppose $|f(z)| \leq M$ for all z. Then

$$|f'(z)| \leq \frac{1}{2\pi} \int_{C_R(z)} \frac{|f(w)|\,|dw|}{|w-z|^{2}} \qquad (\text{let } n=1)$$

$$\leq \frac{1}{2\pi} \int_{C_R(z)} \frac{M}{R^2}\,|dw| = \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R}$$

Since $f(z)$ is entire, we can let $R \to \infty \Rightarrow f'(z) = 0,\ \forall z$.

Thus, $\boxed{f(z) = k}$, a constant.

11. (15 points) Let $f(z) = z^9 + 6z^5 - z^3 + z^2 - 2$.

(a) How many zeroes (counting multiplicity) does $f(z)$ have in all? What theorem do you need here?

By the fundamental theorem of algebra, f has 9 zeros.

(b) How many zeroes (counting multiplicity) does $f(z)$ have inside the circle of radius 1 centered at 0? What theorem do you need for this question?

$f(z) = g(z) + h(z)$ where $g(z) = 6z^5$
and $h(z) = z^9 - z^3 + z^2 - 2$.

Since $|h(z)| < |g(z)|$ on $C_1(0)$, by Rouche's theorem

$$\begin{matrix}\#\text{zeros of } f \\ \text{inside } C_1(0)\end{matrix} = \begin{matrix}\#\text{zeros of } g \\ \text{inside } C_1(0)\end{matrix} = 5.$$

(c) How many zeroes are there inside the circle of radius 1/2?

$f(z) = g(z) + h(z)$ where $g(z) = -2$
and $h(z) = z^9 + 6z^5 - z^3 + z^2$.

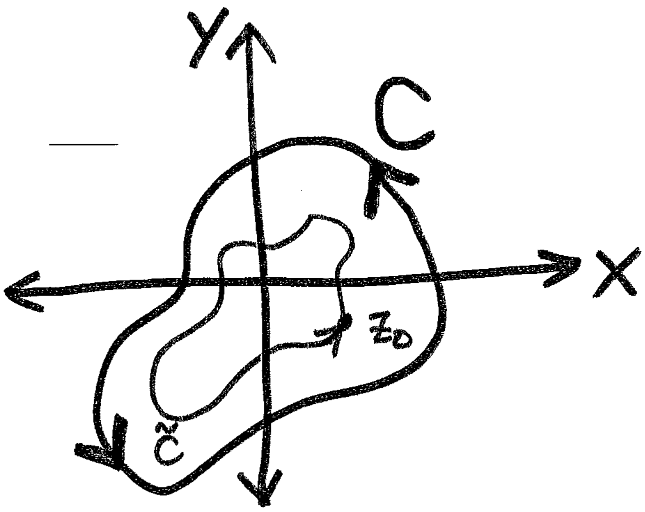
Since $|h(z)| < |g(z)|$ on $C_{1/2}(0)$, by Rouche's theorem

$$\begin{matrix}\#\text{zeros of } f \\ \text{inside } C_{1/2}(0)\end{matrix} = \begin{matrix}\#\text{zeros of } g \\ \text{inside } C_{1/2}(0)\end{matrix} = 0.$$

12. (15 points) This problem is difficult, so finish the rest of the exam before focusing on this problem. Also, since this is a proof problem, credit will only be given for actual proofs, not guesses. This problem also requires some new ideas, so don't hesitate to think out of the box a little bit.

Suppose that $f(z)$ is analytic on and inside a simple closed curve C which bounds a region and goes around the region in the counterclockwise direction, as drawn. Suppose that $|f(z)| = 1$ on C and that $f(z)$ never equals zero inside C.

(a) What is the winding number of $f(C)$ about the origin in the image? Prove your answer.



By the Argument Principle,

$$\begin{array}{c}\text{winding number of} \\ f(C) \text{ about origin}\end{array} = \begin{array}{c}\#\text{zeros of} \\ f \text{ inside } C\end{array} = \boxed{0}$$

since we're given that $f(z)$ never equals zero inside C.

(b) Prove that it is possible to define $g(z) = \log(f(z))$ as a single-valued analytic function defined for z inside C.

$$g(z) = \log(f(z)) = \ln|f(z)| + i\arg(f(z)).$$

Claim: $\arg(f(z))$ can be taken to be single-valued.

Proof: If not, there would exist a path $\tilde{C}$ from some point z_0 back to z_0 (see figure) where $\Delta_{\tilde{C}} \arg(f(z)) \neq 0$. But

$$\Delta_{\tilde{C}} \arg(f(z)) = 2\pi \begin{pmatrix}\text{winding number} \\ \text{of } f(\tilde{C}) \text{ about origin}\end{pmatrix} = 2\pi \begin{pmatrix}\#\text{zeros of} \\ f \text{ inside } \tilde{C}\end{pmatrix} = 0$$ since f has no zeros. ∎

Alternatively, $$g(z) = \left(\int_{z_0}^{z} \frac{f'(z)}{f(z)}\,dz\right) + k$$ which is well-defined since $f(z) \neq 0$ inside C so that $\frac{f'(z)}{f(z)}$ is an analytic function. $(k = \mathrm{Log}\, f(z_0))$

(c) Using the fact that a harmonic function defined inside C is determined uniquely by its values on C, prove that $g(z)$ is constant inside C.

$\mathrm{Re}(g(z)) = \ln|f(z)| = \ln 1 = 0$ on C.

Harmonic and $=0$ on C, so it must be 0 everywhere since 0 is a harmonic function.

By Cauchy-Riemann equations, $\mathrm{Re}(g(z)) = 0$ inside C $\Rightarrow$ $\mathrm{Im}(g(z)) = k$ $\Rightarrow$ $\boxed{g(z) = ik}$, where k is a constant.

(d) Prove that $f(z)$ must be constant inside C.

$$f(z) = e^{\log f(z)} = e^{g(z)} = \boxed{e^{ik}} = \text{constant}.$$

(e) Conversely, given a *nonconstant* function $h(z)$ which is analytic on and inside C which has norm 1 on C, prove that $h(z)$ must have at least one zero inside C.

By (a)-(d), if $h(z)$ does not have any zeros inside C, then $h(z)$ must be a constant. But we're given that $h(z)$ is not a constant. Hence, $h(z)$ must have at least one zero inside C.