Your Name: Key

Midterm

Complex Analysis, Math 333

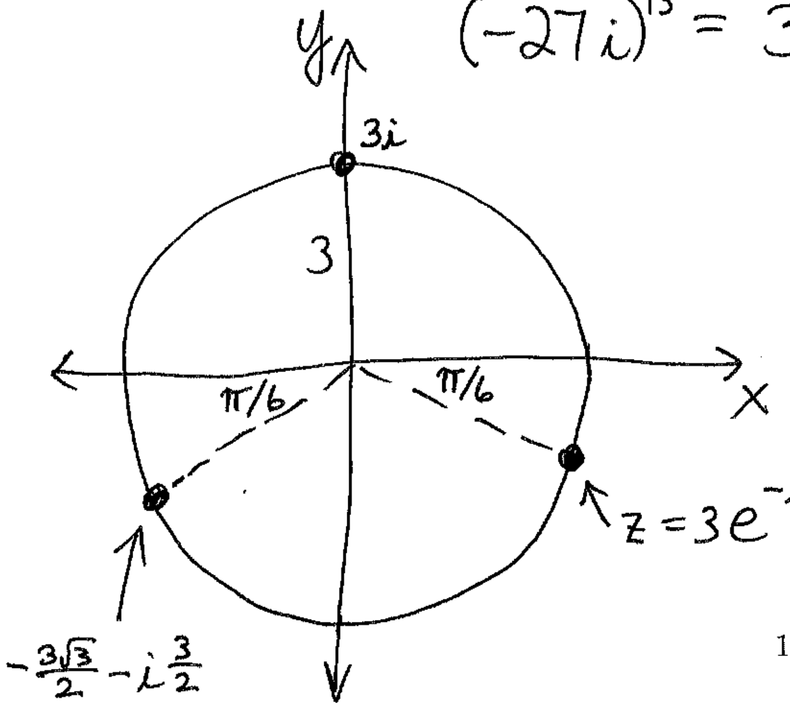
October 8, 2014

Please write your name on this exam in the space provide above. There are 100 points on this 75 minute exam. Show all of your work to get full credit on each problem. You may write on the backs of pages if you need extra space. Good luck!

1. (10 points) Plot the third roots of $-27i$ and express each of the roots in polar coordinates.

$$-27i = 27e^{-i\frac{\pi}{2}} \quad \text{so}$$

$$(-27i)^{1/3} = 3e^{i\left(-\frac{\pi}{6} + \frac{2\pi}{3}k\right)}, \quad k = 0, 1, 2$$



$$z = 3e^{-i\frac{\pi}{6}} = 3\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$

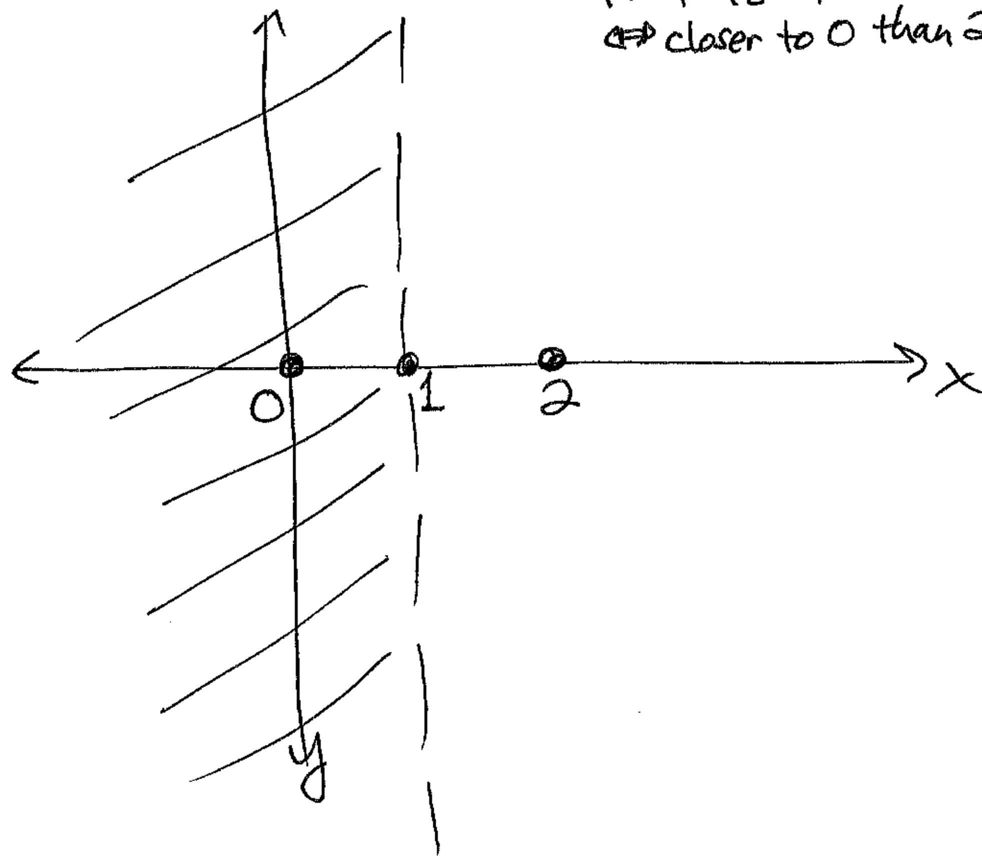
$$= 3\left(\frac{\sqrt{3}}{2} - i\cdot\frac{1}{2}\right)$$

$$= \frac{3\sqrt{3}}{2} - i\frac{3}{2}$$

2. (10 points) Describe and graph the region determined by $|z| < |z - 2|$.

$|z-0| < |z-2|$

$\Leftrightarrow$ closer to 0 than 2



Also, $|z|^2 < |z-2|^2$

$x^2+y^2 < (x-2)^2+y^2$

$x^2 < x^2-4x+4$

$4x < 4$

$\boxed{x<1}$.

3. (10 points)

(a) Suppose $f(z)$ is analytic in the region $|z| < 1$. Suppose that $\overline{f}(z)$ is analytic in this region also. Give some examples of possible functions that $f(z)$ might be, and then prove that this is all of them.

$f(z) = k$, a constant.

$f(z)$ analytic $\Leftrightarrow$ $u_x = v_y$, $u_y = -v_x$
(where $f = u + iv$)

$\overline{f}(z)$ analytic $\Leftrightarrow$ $u_x = -v_y$, $u_y = v_x$
(since $\overline{f} = u - iv$)

$\rightarrow$ $u_x = v_y = 0$

$\rightarrow$ $u_y = v_x = 0$

$f(z) = k$. $\leftarrow$ u, v are constants $\leftarrow$

(b) Suppose that $f(z)$ is analytic in the region $|z| < 1$ and that $|f(z)| = 7$ in this region as well. Without quoting any major theorems (like the Maximum Modulus Theorem) other than the result of part (a) above, solve for every possible function $f(z)$.

$49 = |f(z)|^2 = f(z)\overline{f}(z) \longrightarrow \overline{f}(z) = \frac{49}{f(z)}$ is analytic too

$\downarrow$

Both $f(z), \overline{f}(z)$ are analytic

$f(z) = \text{const.}$ $\xleftarrow{\text{by (a)}}$

$\downarrow$

$\boxed{f(z) = 7e^{i\theta_0}}$ for some fixed θ_0.

since $|f(z)| = 7$.

4. (15 points) Let $f(z) = \frac{\exp(z)}{z^3}$.

(a) Find the Laurent series expansion for $f(z)$ around $z_0 = 0$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

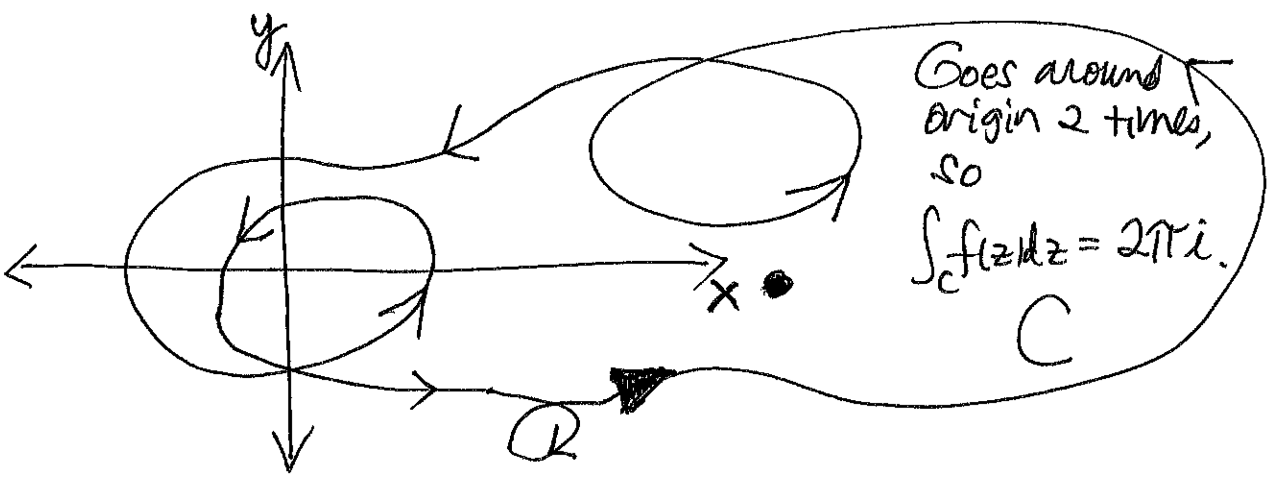
$$f(z) = \frac{e^z}{z^3} = \sum_{k=0}^{\infty} \frac{z^{k-3}}{k!} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!\,z} + \frac{1}{3!} + \frac{z}{4!} + \frac{z^2}{5!} + \dots$$

Note that $b_1 = \frac{1}{2}$.

(b) What is $\int_C f(z)dz$ when C is the circle of radius 2 going counterclockwise around the origin? (Hint: Don't actually do the contour integral. Use part (a).)

$$\int_C f(z)dz = 2\pi i \cdot b_1 = \pi i.$$

(c) What is $\int_C f(z)dz$ when C is the curve drawn below?



Goes around origin 2 times, so $\int_C f(z)dz = 2\pi i$.

5. (15 points) Let C be the circle of radius 100 going counterclockwise around the origin. Find

(a) $\int_C \frac{z^2 dz}{z-5} = \int_C \frac{f(z)dz}{z-5} = 2\pi i f(5) = 50\pi i$

(where $f(z) = z^2$)

(b) $\int_C \frac{z^2 dz}{(z-5)^2} = \int_C \frac{f(z)dz}{(z-5)^2} = 2\pi i f'(5) = 2\pi i \cdot 10 = 20\pi i$.

$\left(\begin{array}{l} f(z) = z^2 \\ f'(z) = 2z \end{array} \right)$

(b) $\int_C \frac{z^2 dz}{(z-500)^3} = 0$ ~~since~~ by Cauchy-Goursat since $\frac{z^2}{(z-500)^3}$ is analytic in $B_{100}(0)$.

6. (15 points)

(a) Determine the Taylor series for $f(z) = \frac{4z^2}{1-4z^2}$ around $z_0 = 0$.

$$f(z) = \frac{1}{1-4z^2} - 1 = 4z^2 + (4z^2)^2 + (4z^2)^3 + \ldots$$

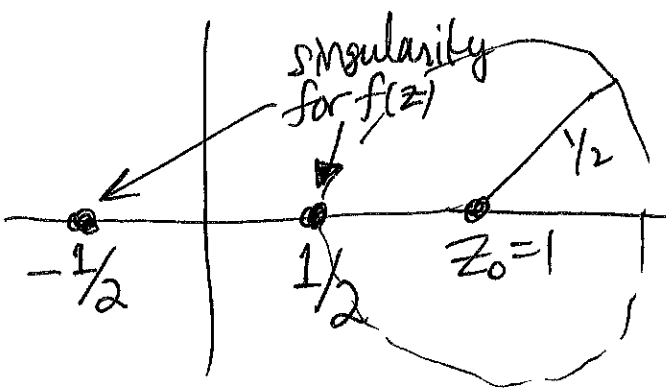
$$= \sum_{k=1}^{\infty} (4z^2)^k = \sum_{k=1}^{\infty} 4^k z^{2k}.$$

$$\text{for } |4z^2| < 1 \Leftrightarrow |z| < \tfrac{1}{2}.$$

(b) What is the radius of convergence of this Taylor series?

$1/2$.

(c) What is the radius of convergence of the Taylor series for this same $f(z)$ around $z_0 = 1$? Note that you do not have to compute the Taylor series itself, just determine its radius of convergence.



$\frac{1}{2}$ since $z_0 = 1$ is $\frac{1}{2}$ away from the singularity at $1/2$.

7. (10 points)

(a) State and prove Liouville's Theorem (about bounded analytic functions defined on the entire complex plane).

Cauchy Integral Formula for $f'(z)$: $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}$

Let C be the circle of radius R around z. Then

$$|f'(z)| \le \left|\frac{1}{2\pi i}\right| \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R} \to 0 \text{ as } R \to \infty.$$

$$\longrightarrow f'(z) = 0 \longrightarrow f(z) = k, \text{ a constant.}$$

Liouville's Thm: If $f(z)$ is entire and $|f(z)| < M$, then $f(z) = k$, a constant.

(b) Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_0$, for some $n \geq 1$ and $a_n \neq 0$. Prove that there exists a value of $z \in C$ such that $P(z) = 0$.

Suppose not. Then

$$f(z) = \frac{1}{P(z)} = \frac{1}{z^n\left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right)}$$ is analytic.

goes to 0 at ∞

Since $f(z) \to 0$ at ∞, $f(z)$ is bounded.

Hence, by Liouville's Theorem, $f(z) = k \Rightarrow P(z) = \frac{1}{k}$.

Contradiction.

Hence, $\exists z$ where $P(z) = 0$.

8. (15 points) Generalization of Liouville's Theorem. Suppose that $f(z)$ is entire (analytic in the complex plane) and that

$$|f(z)| \leq A|z|^3 + B$$

for all z and for some real-valued positive constants A and B.

(a) Give an example of a nonconstant analytic function which satisfies this bound for some A and B.

$$f(z) = z^3 \qquad (A=1, B=0)$$

(b) Make a conjecture about the set of all entire functions $f(z)$ which satisfy the above bound.

$$\{az^3 + bz^2 + cz + d\} \quad \text{(all cubics)}.$$

(c) Prove your conjecture.

Cauchy Integral Formula for $f^{(4)}(z) = \dfrac{4!}{2\pi i} \displaystyle\int_{C_R(z)} \frac{f(s)\,ds}{(s-z)^5}$.

$$|f^{(4)}(z)| \leq \frac{4!}{|2\pi i|} \cdot \int_{C_R(z)} \frac{A|s|^3 + B}{R^5} |ds| \leq \frac{4!}{2\pi} \int_{C_R} \frac{A(|z|+R)^3 + B}{R^5} |ds|$$

$$= \frac{4!}{2\pi} \left(\frac{A(|z|+R)^3 + B}{R^5} \right) 2\pi R$$

$$= 4! \, \frac{A(|z|+R)^3 + B}{R^4}$$

$$\to 0 \text{ as } R \to \infty.$$

Hence, $f^{(4)}(z) = 0$

$$\Downarrow$$

$$f^{(3)}(z) = a$$

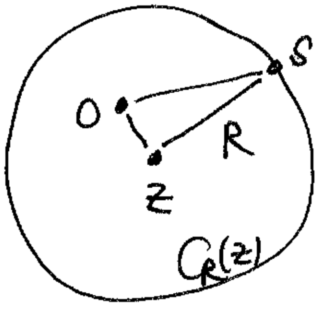
$$\Downarrow$$

$$f^{(2)}(z) = az + b$$

$$\Downarrow$$

$$f'(z) = \frac{a}{2} z^2 + bz + c$$

$$\boxed{f(z) = \frac{a}{6} z^3 + \frac{b}{2} z^2 + cz + d}$$



Δ ineq:
$|s| \leq |z| + R$