Exam #3 Math 108 – Fall 2007

Honor pledge: I have adhered to the Duke Community standard in completing this test.

Name:

Score:

1. Consider the numbers $a_n = 2\int_0^1 \sqrt{x}\cos(n\pi x)dx$, for $n = 0, 1, 2, 3, \ldots$

(a) (10 pts.) Sketch the graph of the function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

over the interval $[-2, 2]$.

(b) (5 pts.) What is the value of $f(4)$?

2. (10 pts.) Find all the eigenvalues λ_n and the corresponding eigenfunctions X_n for the problem

$$X'' + \lambda X = 0$$
$$X(0) = 0, \ X'(\pi) = 0.$$

3. (20 pts.) Solve the partial differential equation

$$\begin{aligned} &u_{xx} = u_t, \\ &u(0,t) = 0, \ \ u_x(\pi,t) = 0, \\ &u(x,0) = \sin\frac{3x}{2} + 2\sin\frac{7x}{2}, \end{aligned}$$

where $0 < x < \pi$ and $t > 0$.

DO NOT INCLUDE INTEGRALS IN YOUR FINAL ANSWER.

HINT: You may want to use the result of Problem 2 for some parts.

(Extra space for problem 3: $u_{xx} = u_t,\ u(0,t) = 0,\ u_x(\pi,t) = 0,\ u(x,0) = \sin(3x/2) + 2\sin(7x/2)$.)

4. (10 pts.) Find the **steady state** solution of the following problem:

$$u_t = u_{xx} - u$$
$$u(0,t) = 0$$
$$u_x(L,t) = 2.$$

Here $0 < x < L,\ t > 0$ and T is a positive constant.

5. (15 pts.) Find the solution of the Dirichlet problem

$$u_{xx} + u_{yy} = 0$$

with the boundary conditions

$$u(x,0) = 0,\ u(x,1) = 0,\ 0 < x < 1$$
$$u(0,y) = 0,\ u(1,y) = 2,\ 0 \leq y \leq 1.$$

(You may use a formula from class.)

Do not include integrals in your final answer.

(Extra space for problem 5: $u_{xx} + u_{yy} = 0,\ u(x, 0) = 0,\ u(x, 1) = 0,\ u(0, y) = 0,\ u(1, y) = 2.$)

6. The dispersive wave equation is given by

$$\frac{1}{a^2}u_{tt} + \gamma^2 u = u_{xx}, \; 0 < x < L, t > 0$$

with the boundary conditions

$$u(0,t) = 0, \; u(L,t) = 0, \; t > 0$$

and the initial conditions

$$u(x,0) = f(x), \; u_t(x,0) = 0, \; t > 0.$$

(a) (10 pts.) **Step 1.** Using separation of variables find the eigenvalue problems for this equation.

(Problem 6 continues...)

(b) (10 pts.) **Step 2.** Solve the eigenvalue problems you found in (a).

(Problem 6 continues...)

(c) (10 pts.) **Step 3.** Deduce using Fourier series that the solution of the dispersive wave equation has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(at\sqrt{\frac{n^2\pi^2}{L^2} + \gamma^2}\right) sin\frac{n\pi x}{L},$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$