Exam 2
Math 353
Summer Term I, 2014
Friday, June 13, 2014
Time Limit: 75 Minutes

Name: Key

This exam contains 6 pages (including this cover page) and 6 questions, plus a table of Laplace transforms at the very end. The total number of points on this exam is 72.

Your are allowed to use a calculator on this exam, though it is not really necessary. While this is a closed book exam, you are allowed to use your one page review sheet, front and back, written in your own handwriting.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
Total:	72	

1. (12 points) Find the inverse Laplace transforms of

(a) $F(s) = \frac{3}{s^2+4} = \frac{3}{2} \cdot \frac{2}{s^2+2^2}$

$\mathcal{L}^{-1}\{F(s)\} = f(t) = \boxed{\frac{3}{2}\sin 2t}$

(b) $F(s) = \frac{2}{s^2+3s-4} = \frac{2}{(s-1)(s+4)} = \frac{a}{s+4} + \frac{b}{s-1} = \frac{a(s-1)+b(s+4)}{(s+4)(s-1)}$

$= \frac{(a+b)s + (4b-a)}{(s+4)(s-1)}$

$\left.\begin{array}{l} a+b=0 \\ 4b-a=2 \end{array}\right\} \begin{array}{l} a=-\frac{2}{5} \\ b=\frac{2}{5} \end{array}$

$\mathcal{L}^{-1}\{F(s)\} = f(t) = \boxed{\frac{2}{5}e^{t} - \frac{2}{5}e^{-4t}}$ $\left(= e^{-(3/2)t} \cdot \frac{4}{5}\sinh\left(\frac{5}{2}t\right) \text{ also}\right)$

2. (12 points) Using the Laplace transform, find the solution to the initial value problem

$$y'' + 4y = \sin(t) - u_{2\pi}(t)\sin(t - 2\pi), \qquad y(0) = 0, \qquad y'(0) = 0,$$

where $u_c(t)$ is the unit step function equal to 1 for $t \geq c$ and 0 otherwise.

$$(s^2+4)Y(s) = \frac{1}{s^2+1}\left(1 - e^{-2\pi s}\right)$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)}\left(1 - e^{-2\pi s}\right)$$

$$\frac{1}{s^2+1} - \frac{1}{s^2+4} = \frac{(s^2+4)-(s^2+1)}{(s^2+1)(s^2+4)} = \frac{3}{(s^2+1)(s^2+4)}$$

$$Y(s) = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)\left(1 - e^{-2\pi s}\right)$$

$$= H(s)\left(1 - e^{-2\pi s}\right)$$

where $H(s) = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{2}{s^2+4}\cdot\frac{1}{2}\right) = \mathcal{L}\left\{\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right\}$

Thus,

$$\boxed{\begin{aligned} &y(t) = h(t) - u_{2\pi}(t)h(t-2\pi) \quad \text{where} \\ &h(t) = \tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t \end{aligned}}$$

3. (12 points) Find the solution to the initial value problem

$$y'' + 2y' + 2y = \delta(t - \pi), \qquad y(0) = 1, \qquad y'(0) = 0,$$

where δ is the Dirac delta function.

$$(s^2 Y(s) - s) + 2(sY(s) - 1) + 2Y(s) = e^{-\pi s}$$

$$(s^2 + 2s + 2)Y(s) - (s+2) = e^{-\pi s}$$

$$Y(s) = \frac{e^{-\pi s} + s + 2}{(s+1)^2 + 1} = \frac{e^{-\pi s} + (s+1) + 1}{(s+1)^2 + 1}$$

$$= \frac{e^{-\pi s}}{(s+1)^2 + 1} + \frac{s+1}{(s+1)^2 + 1} + \frac{1}{(s+1)^2 + 1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \boxed{u_\pi(t) e^{-(t-\pi)} \sin(t-\pi) + e^{-t}\cos(t) + e^{-t}\sin(t}$$

4. (12 points) Let

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x), \qquad \text{where} \qquad b_n = 2\int_0^1 x^2 \sin(n\pi x)dx.$$

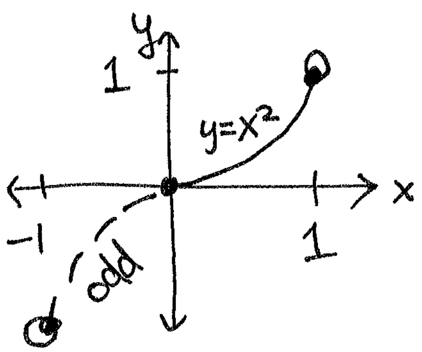
(a) Is $f(x)$ an even function, an odd function, or neither?

$\boxed{\text{odd}}$, since $\sin(x)$ is odd.

(b) What is the period of $f(x)$?

$\boxed{2}$, since $f(x+2) = \sum_{n=1}^{\infty} b_n \sin(n\pi x + 2\pi n) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$
$= f(x)$.

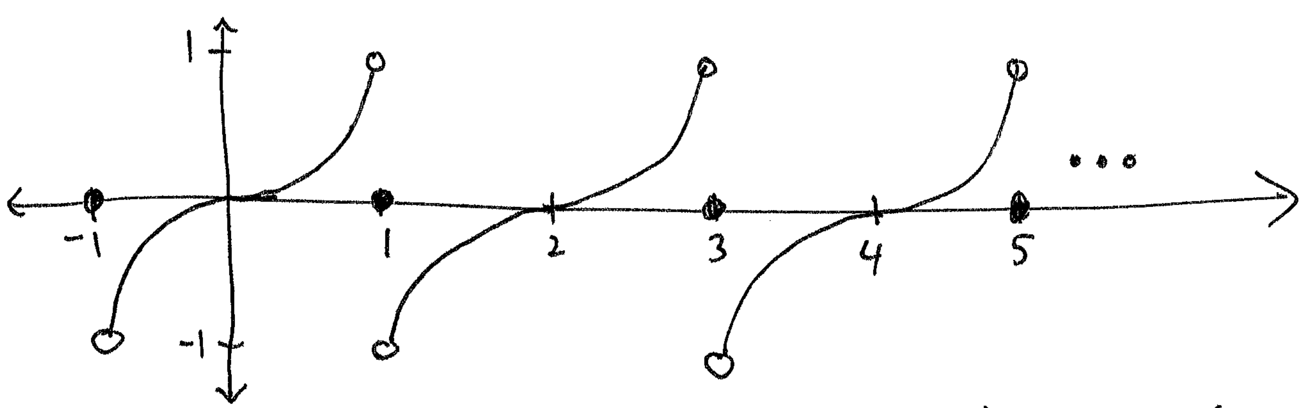
(c) Graph $f(x)$ for $0 \le x \le 1$.



by Fourier series theorem.

(d) What is $f(1.5)$?

(Hint: Using (a) and (b), consider what the graph of $f(x)$ for all x looks like.)



$$f(1.5) \underset{\text{period } 2}{=} f(1.5-2) = f(-0.5) \underset{\text{odd}}{=} -f(0.5) = -\left(\frac{1}{2}\right)^2$$

$$= \boxed{-\frac{1}{4}}$$

5. (12 points) Consider the heat conduction problem

$$u_{xx} = 4u_t, \qquad 0 < x < 2, \qquad t > 0$$

$$u(0,t) = 0, \qquad u(2,t) = 0, \qquad t > 0$$

$$u(x,0) = \underset{n=1}{2\sin(\pi x/2)} - \underset{n=2}{\sin(\pi x)} + \underset{n=4}{4\sin(2\pi x)}$$

(a) Find the solution $u(x,t)$.

$$\boxed{u_n = \sin\left(n\frac{\pi}{2}x\right) e^{-\left(\frac{n\pi}{4}\right)^2 t}} \longrightarrow$$

$$\boxed{u(x,t) = 2u_1(x,t) - u_2(x,t) + 4u_4(x,t)}$$

$$= 2\sin\left(\frac{\pi x}{2}\right)e^{-\frac{\pi^2}{16}t} - \sin(\pi x)e^{-\frac{\pi^2}{4}t} + 4\sin(2\pi x)e^{-\pi^2 t}$$

(b) What is the steady state solution as t goes to infinity?

$$\boxed{\lim_{t\to\infty} u(x,t) = 0}$$

6. (12 points) Find the solution $u(x,t)$ to the wave equation problem

$$4u_{xx} = u_{tt}, \qquad 0 < x < \pi, \qquad t > 0$$

$$u(0,t) = 0, \qquad u(\pi,t) = 0, \qquad t > 0$$

where

$$\begin{aligned} u(x,0) &= 0 \\ u_t(x,0) &= 3\sin(x). = g(x) \end{aligned}$$

for $0 \le x \le \pi$.

Let $u(x,t) = X(x)T(t)$. $\rightarrow \dfrac{X''}{X} = \dfrac{T''}{4T} = -\lambda \longrightarrow$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0,\ X(\pi) = 0 \end{cases} \quad \text{and} \quad \begin{cases} T'' + 4\lambda T = 0 \\ T(0) = 0 \end{cases}$$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$X_n = \sin(nx)$ $\qquad$ $T(t) = \sin(2nt)$

$\lambda_n = n^2$

$\therefore\ u_n(x,t) = \sin(nx)\sin(2nt)$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx)\sin(2nt)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} (2nC_n)\sin(nx) \longrightarrow \begin{array}{l} C_n = 0 \text{ for all } n \neq 1 \\ C_1 = 3/2 \end{array}$$

$$\boxed{u(x,t) = \frac{3}{2}\sin(x)\sin(2t)}$$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28