**Final Exam**
**Math 353**
**Summer Term I, 2014**
**Thursday, June 26, 2014**
**Time Limit: 3 hours**

**Name:** Key

This exam contains 11 pages (including this cover page) and 10 questions, plus a table of Laplace transforms at the very end. The total number of points on this exam is 120.

Your are allowed to use a calculator on this exam, though it is not really necessary. While this is a closed book exam, you are allowed to use your one page review sheet, front and back, written in your own handwriting.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

| Question | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 12 | |
| 3 | 12 | |
| 4 | 12 | |
| 5 | 12 | |
| 6 | 12 | |
| 7 | 12 | |
| 8 | 12 | |
| 9 | 12 | |
| 10 | 12 | |
| Total: | 120 | |

1. (12 points) Find the inverse Laplace transforms of

(a) $F(s) = \frac{e^{-s}}{3s}$

$$\boxed{f(t) = \frac{1}{3} u_1(t)}$$

(b) $F(s) = \frac{7}{s^2-4s+5}$ $= \frac{7}{(s-2)^2+1}$

$$\boxed{f(t) = 7e^{2t} \sin(t)}$$

(c) $F(s) = \frac{6s^2}{s^4-1}$ $= 3\left(\frac{1}{s^2+1} + \frac{1}{s^2-1}\right)$

$$\boxed{f(t) = 3\sin(t) + 3\sinh(t)}$$

2. (12 points) Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0,$$

where $\delta$ is the Dirac delta function.

$$(s^2+4)\,Y(s) = e^{-\pi s} - e^{-2\pi s}$$

$$Y(s) = \frac{1}{2} \cdot \frac{2}{s^2+4}\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$y(t) = \frac{1}{2} u_\pi(t) h(t-\pi) - \frac{1}{2} u_{2\pi}(t) h(t-2\pi)$$

where $h(t) = \sin(2t) \longrightarrow$

$$y(t) = \frac{1}{2} u_\pi(t) \sin 2t - \frac{1}{2} u_{2\pi}(t) \sin(2t)$$

$$= \boxed{\frac{1}{2} \sin 2t \left(u_\pi(t) - u_{2\pi}(t)\right)}$$

3. (12 points) Find the general solutions to the following differential equations.

(a) $ty' + 2y = t^6$

$$t^2y' + (2t)y = t^7$$

$$(t^2y)' = t^7$$

$$t^2y = \frac{1}{8}t^8 + c$$

$$\boxed{y = \frac{1}{8}t^6 + \frac{c}{t^2}}$$

(b) $y'' - 4y' + 5y = t$

$$r^2 - 4r + 5 = 0 \longrightarrow$$

$$r = \frac{4 \pm \sqrt{16-20}}{2}$$

$$r = 2 \pm i$$

$$y_H = e^{2t}(c_1 \cos t + c_2 \sin t)$$

$$y_P = \frac{t}{5} + \frac{4}{25} \quad \text{(Guess } y_P = a + bt\text{, solve for } a, b.\text{)}$$

$$\boxed{y = y_H + y_P = e^{2t}(c_1 \cos t + c_2 \sin t) + \frac{t}{5} + \frac{4}{25}}$$

4. (12 points) Find the general solutions to the following differential equations.

(a) $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$

$$(2y-2)\,dy = (3x^2+4x+2)\,dx$$

$$\boxed{y^2-2y = x^3+2x^2+2x+C}.$$ Also,

$$(y-1)^2 = y^2-2y+1 = x^3+2x^2+2x+K$$

$$\boxed{(y-1)^2 = x^3+2x^2+2x+K}.$$

(b) $\frac{dy}{dx} = \frac{y-4x}{x-y}$ (Hint: Let $v = \frac{y}{x}$ and transform this to an equation for $v(x)$.)

$$y = v\cdot x$$

$$\frac{dy}{dx} = \frac{dv}{dx}\cdot x + v$$

$$\frac{dv}{dx}\cdot x + v = \frac{v-4}{1-v}$$

$$x\frac{dv}{dx} = \frac{v-4}{1-v} - v\frac{1-v}{1-v} = \frac{v^2-4}{1-v}$$

$$\frac{dx}{x} = \frac{1-v}{v^2-4}\,dv = \left(-\frac{3}{4}\cdot\frac{1}{v+2} - \frac{1}{4}\cdot\frac{1}{v-2}\right)dv$$

$$\boxed{\ln|x| = -\frac{3}{4}\ln|v+2| - \frac{1}{4}\ln|v-2| + C}$$ Also

$$Kx = (v+2)^{-3/4}(v-2)^{-1/4}$$

$$K = (vx+2x)^{-3/4}(vx-2x)^{-1/4}$$

$$K = (y+2x)^{-3/4}(y-2x)^{-1/4} \longrightarrow \boxed{(y+2x)^3(y-2x) = \tilde{K}}$$

5. (12 points) Consider the first order differential equation

$$\underbrace{(3x^2y\sin y + y^2e^{xy})}_{M} + \underbrace{(x^3y\cos y + xye^{xy})}_{N}\frac{dy}{dx} = 0.$$

(a) Find an integrating factor which makes the above differential equation exact. (Hint: The integrating factor is a function of $y$.)

From book p.99 and #23, p.100:

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M} \cdot \mu = \ldots = -\frac{1}{y}\mu$$

$$\rightarrow \frac{d\mu}{\mu} = -\frac{dy}{y}$$

$$\rightarrow \boxed{\mu(y) = \frac{1}{y}}$$

(b) Find the solution to the differential equation with $y(1) = 1$. (An implicitly defined solution is fine.)

$$(3x^2\sin y + ye^{xy}) + (x^3\cos y + xe^{xy})\frac{dy}{dx} = 0$$

$$\downarrow$$

$$\frac{d}{dx}(x^3\sin y + e^{xy}) = 0$$

$$\downarrow$$

$$\boxed{x^3\sin y + e^{xy} = \text{const} = \sin(1) + e}$$

6. (12 points) By expanding in a power series around $x_0 = 0$, find an everywhere analytic solution to

$$x^2y'' + y' - 2y = -1,$$

with initial conditions $y(0) = 1$ and $y'(0) = 1$. Compute the first 5 terms of the power series solution and express the solution $y(x)$ as simply as possible. (Hint: Consider the constant terms on both sides of the equation separately from the rest of the power series expansion.)

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$x^2 \cdot y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$$\rightarrow \sum_{n=0}^{\infty} \left[n(n-1)a_n + (n+1)a_{n+1} - 2a_n\right]x^n = -1$$

$$(n^2-n-2)a_n + (n+1)a_{n+1}$$

$n=0$: $\boxed{a_1 - 2a_0 = -1}$ ✓

$n \geq 1$: $\boxed{a_{n+1}} = -\frac{n^2-n-2}{n+1}a_n = \boxed{-\frac{(n-2)\cancel{(n+1)}}{\cancel{n+1}}a_n}$

$\boxed{a_{n+1} = (2-n)a_n}$

$n=1$: $\boxed{a_2 = a_1}$

$n=2$: $a_3 = 0$ !

$n=3$: $a_4 = 0$ !

$n=4$: $a_5 = 0$ !

$\vdots$

In our case:

$a_0 = 1$, $a_1 = 1$ } given

$a_2 = 1$

$a_n = 0$, $n \geq 3$

$$\boxed{y = 1 + x + x^2}$$

$\left(y = -x - x^2 + a_0(1 + 2x + 2x^2)\right)$

7. (12 points) Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x), \qquad \text{where} \qquad a_n = 2\int_0^1 x\cos(n\pi x)\,dx.$$

(a) Is $f(x)$ an even function, an odd function, or neither?

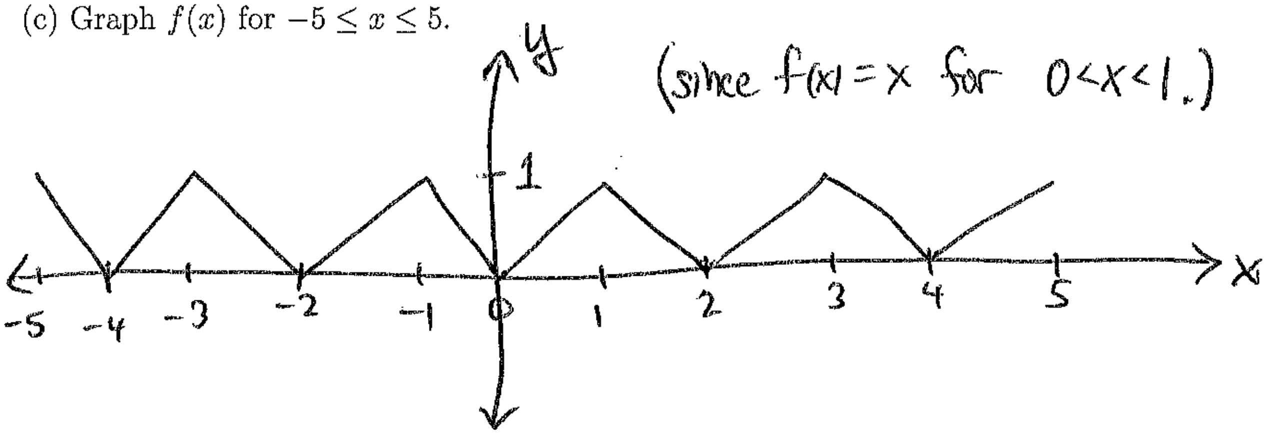
$\boxed{\text{even}}$, since $\cos(x)$ is even.

(b) What is the period of $f(x)$?

$T = \boxed{2}$ since $f(x+2) = f(x)$.

(c) Graph $f(x)$ for $-5 \le x \le 5$.

(since $f(x) = x$ for $0 < x < 1$.)




(d) What is $f(1.5)$?

$f(1.5) = f(1.5-2) = f(-0.5) = f(0.5) = \boxed{0.5}$.

8. (12 points) Consider the heat conduction problem

$$9u_{xx} = u_t, \qquad 0 < x < 1, \qquad t > 0$$

$$u_x(0,t) = 0, \qquad u_x(1,t) = 0, \qquad t > 0$$

$$u(x,0) = 2\cos(\pi x) - \cos(2\pi x) + 3$$

$U_n(x,t) = X_n(x)T_n(t)$

(a) Find the solution $u(x,t)$.

$$\frac{X''}{X} = \frac{T'}{9T} = -\lambda \longrightarrow \begin{cases} X'' + \lambda X = 0, \\ X'(0) = 0 \\ X'(1) = 0 \end{cases} \quad \begin{cases} T' + 9\lambda T = 0 \\ T(0) = 1 \end{cases}$$

$X_n(x) = \cos(n\pi x)$

$\lambda_n = n^2\pi^2$

$T_n(x) = e^{-9n^2\pi^2 t}$

$\downarrow$

$U_n(x,t) = e^{-9n^2\pi^2 t}\cos(n\pi x)$

$$\boxed{u(x,t) = 3 + 2e^{-9\pi^2 t}\cos(\pi x) - e^{-36\pi^2 t}\cos(2\pi x)}$$

(b) What is the steady state solution as $t$ goes to infinity?

$$\lim_{t\to\infty} u(x,t) = \boxed{3}.$$

9. (12 points) Find the solution $u(x,t)$ to the wave equation problem

$$u_{xx} = u_{tt}, \qquad 0 < x < \pi, \qquad t > 0$$

$$u(0,t) = 0, \qquad u(\pi,t) = 0, \qquad t > 0$$

where

$$\begin{aligned} u(x,0) &= 5\sin(2x) \\ u_t(x,0) &= 3\sin(x). \end{aligned}$$

for $0 \le x \le \pi$. (Hint: Break this up into two separate problems.)

Problem 1:

$u(x,0) = 5\sin 2x$

$u_t(x,0) = 0$

$\downarrow \cdots$

$u_1(x,t) = 5\sin(2x)\underline{\cos(2t)}$

Problem 2

$u(x,0) = 0$

$u_t(x,0) = 3\sin(x)$

$\downarrow \cdots$

$u_2(x,t) = 3\sin(x)\underline{\sin(t)}$

Add together: $\boxed{u(x,t) = 5\sin 2x \cos 2t + 3\sin x \sin t}$

Both problems have $u_n(x,t) = X(x)T(t)$ where

$X'' + \lambda X = 0$, $T'' + \lambda T = 0$

$\downarrow$

$X(x) = \sin(nx)$

$\downarrow$

$\lambda_n = n^2$

1. $T'(0) = 0 \longrightarrow T(t) = \cos(nt)$
2. $T(0) = 0 \longrightarrow T(t) = \sin(nt)$

which is why the underlined terms above are different.

10. (12 points) **Bonus Problem**

Suppose that $\phi_1(x)$ and $\phi_2(x)$ are two eigenfunctions of the Sturm-Liouville problem

$$[e^x y']' - f(x)y + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0,$$

for some fixed $f(x)$, with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. Prove that

$$\int_0^1 \phi_1(x)\phi_2(x)\,dx = 0$$

directly by using the above given information, WITHOUT quoting Lagrange's identity or any theorems from the book. You may, however, find that the proof of Lagrange's identity is very relevant to the above problem. (Hint: Your answer should have two integrations by parts in it.)

$$\underline{2\int_0^1 \phi_1(x)\phi_2(x)dx} = \int_0^1 \phi_1(x)\cdot(\lambda_2\phi_2)\,dx$$

$$= \int_0^1 \phi_1(x)\left(-[e^x\phi_2']' + f(x)\phi_2\right)dx$$

$$= \int_0^1 \{\phi_1'(x)[e^x\phi_2'(x)] + \phi_1(x)f(x)\phi_2(x)\}\,dx - \phi_1 e^x \phi_2'\Big]_0^1$$

0

$$= \int_0^1 \{[e^x\phi_1']\phi_2' + f(x)\phi_1\phi_2\}\,dx$$

0

$$= \int_0^1 \{-[e^x\phi_1']'\phi_2 + f(x)\phi_1\cdot\phi_2\}\,dx + e^x\phi_1'\cdot\phi_2\Big]_0^1$$

$$= \int_0^1 (\lambda_1\phi_1)\phi_2\,dx$$

$$= \underline{\int_0^1 \phi_1(x)\phi_2(x)\,dx} \longrightarrow \boxed{\int_0^1 \phi_1(x)\phi_2(x)\,dx = 0}$$

subtract from both sides ⇒

**TABLE 6.2.1** Elementary Laplace Transforms

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ | Notes |
|---|---|---|
| 1. $1$ | $\dfrac{1}{s}, \quad s > 0$ | Sec. 6.1; Ex. 4 |
| 2. $e^{at}$ | $\dfrac{1}{s-a}, \quad s > a$ | Sec. 6.1; Ex. 5 |
| 3. $t^n, \quad n =$ positive integer | $\dfrac{n!}{s^{n+1}}, \quad s > 0$ | Sec. 6.1; Prob. 27 |
| 4. $t^p, \quad p > -1$ | $\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$ | Sec. 6.1; Prob. 27 |
| 5. $\sin at$ | $\dfrac{a}{s^2+a^2}, \quad s > 0$ | Sec. 6.1; Ex. 7 |
| 6. $\cos at$ | $\dfrac{s}{s^2+a^2}, \quad s > 0$ | Sec. 6.1; Prob. 6 |
| 7. $\sinh at$ | $\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$ | Sec. 6.1; Prob. 8 |
| 8. $\cosh at$ | $\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$ | Sec. 6.1; Prob. 7 |
| 9. $e^{at}\sin bt$ | $\dfrac{b}{(s-a)^2+b^2}, \quad s > a$ | Sec. 6.1; Prob. 13 |
| 10. $e^{at}\cos bt$ | $\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$ | Sec. 6.1; Prob. 14 |
| 11. $t^n e^{at}, \quad n =$ positive integer | $\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$ | Sec. 6.1; Prob. 18 |
| 12. $u_c(t)$ | $\dfrac{e^{-cs}}{s}, \quad s > 0$ | Sec. 6.3 |
| 13. $u_c(t)f(t-c)$ | $e^{-cs}F(s)$ | Sec. 6.3 |
| 14. $e^{ct}f(t)$ | $F(s-c)$ | Sec. 6.3 |
| 15. $f(ct)$ | $\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$ | Sec. 6.3; Prob. 25 |
| 16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$ | $F(s)G(s)$ | Sec. 6.6 |
| 17. $\delta(t-c)$ | $e^{-cs}$ | Sec. 6.5 |
| 18. $f^{(n)}(t)$ | $s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$ | Sec. 6.2 |
| 19. $(-t)^n f(t)$ | $F^{(n)}(s)$ | Sec. 6.2; Prob. 28 |