Final Exam, Math 421
Differential Geometry: Curves and Surfaces in $\mathbf{R}^3$

Instructor: Hubert L. Bray

Tuesday, April 28, 2015

Your Name: Key

Honor Pledge Signature:

Instructions: This is a 3 hour, closed book exam. You may bring one $8\frac{1}{2}'' \times 11''$ piece of paper with anything you like written on it to use during the exam, but nothing else. No collaboration on this exam is allowed. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points. Good luck on the exam!

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
10	12	
Total	~~100~~ 120	

Problem 1. Consider the curve parametrized by

$$\alpha(t) = (\cos(3t), \sin(3t), 4t).$$

(a) What is the speed of α?

$$\alpha'(t) = (-3\sin 3t, 3\cos 3t, 4)$$

$$\text{speed} = |\alpha'(t)| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 16} = \sqrt{25} = 5$$

(b) Find a *unit speed* reparametrization $\beta(s)$.

$$s = 5t \rightarrow t = s/5 \rightarrow$$

$$\beta(s) = \left(\cos\frac{3s}{5}, \sin\frac{3s}{5}, \frac{4s}{5}\right)$$

(c) Using the unit speed reparametrization $\beta(s)$, compute the curvature κ of the curve.

$$\vec{T} = \beta'(s) = \left(-\frac{3}{5}\sin\frac{3s}{5}, \frac{3}{5}\cos\frac{3s}{5}, \frac{4}{5}\right)$$

$$\kappa\vec{N} = \beta''(s) = \left(-\frac{9}{25}\cos\frac{3s}{5}, -\frac{9}{25}\sin\frac{3s}{5}, 0\right)$$

$$\kappa = |\beta''(s)| = \frac{9}{25}$$

(d) Compute all three vectors of the Frenet frame (T, N, B) for the curve $\beta(s)$.

From (c), $\vec{T} = \left(-\frac{3}{5}\sin\frac{3s}{5}, \frac{3}{5}\cos\frac{3s}{5}, \frac{4}{5}\right)$

$\vec{N} = \left(-\cos\frac{3s}{5}, -\sin\frac{3s}{5}, 0\right)$

$\vec{B} = \vec{T}\times\vec{N} = \left(\frac{4}{5}\sin\frac{3s}{5}, -\frac{4}{5}\cos\frac{3s}{5}, \frac{3}{5}\right)$

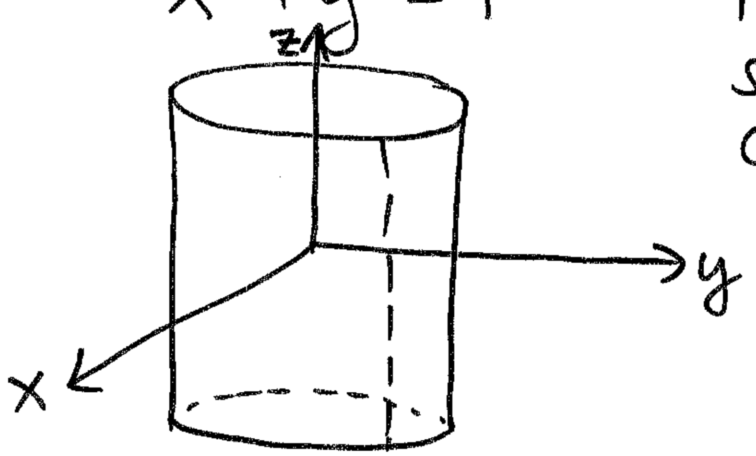
(e) Compute the torsion τ of the curve $\beta(s)$.

$\tau = \vec{N}'(s)\cdot\vec{B} = \left(\frac{3}{5}\sin\frac{3s}{5}, -\frac{3}{5}\cos\frac{3s}{5}, 0\right)\cdot\vec{B}$

$= \frac{12}{25}$

(f) Give the equation for a surface on which this curve is a geodesic. Explain in words why this curve is a geodesic on this surface.

$x^2+y^2=1$



This curve becomes a straight line when this cylinder is cut - - - and flattened in the xy plane.

Problem 2. Define a geodesic of a surface M to be any curve $\alpha(t)$ on M such that $\alpha''(t)$ is perpendicular to M.

(a) Prove that a geodesic has constant speed.

$$\frac{d}{dt}|\alpha'(t)|^2 = \frac{d}{dt}\alpha'(t)\cdot\alpha'(t) = 2\alpha'(t)\cdot\alpha''(t) = 0.$$

Hence, $v = |\alpha'(t)|$ is a constant.

(b) Give the definition of geodesic curvature for a general curve on a surface.

For a unit speed curve, $\kappa_g = \alpha''(t)\cdot(U\times T)$. $\swarrow$ $J(T)$ more generally, the $90°$ rotation of T.

More generally, $v^2\kappa_g = \alpha''(t)\cdot J(T)$, where v is the speed.

(c) Prove that a geodesic has zero geodesic curvature.

is perpendicular to M and hence, so

$v^2\kappa_g = 0 \rightarrow \kappa_g = 0$.

(d) Suppose α is a geodesic on the standard unit sphere. Prove that it is contained in a plane.

$\alpha''(t) \perp$ Sphere $\longrightarrow$ $N \parallel U \rightarrow N = \pm U$

But on the unit sphere, $U(\alpha(t)) = \alpha(t)$,

so

$N'(t) = \pm U'(t) = \pm\alpha'(t) = \pm\vec{T}$ equal

But $N'(t) = \ldots\ldots = -\kappa\vec{T} + \tau\vec{B}$

by the Frenet formulas, so $\tau = 0$.

~~Hence~~ (and $\kappa = 1$). Hence, α is contained in a plane.

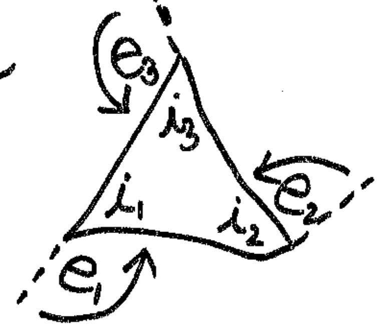
Problem 3.

(a) State the "Gauss Bonnet Theorem for a Disk."

$$\iint_D K\,dA + \int_{\partial D} \kappa_g\,ds = 2\pi$$

(b) State the "Gauss Bonnet Theorem for a Disk with Corners" both in terms of exterior angles and interior angles.

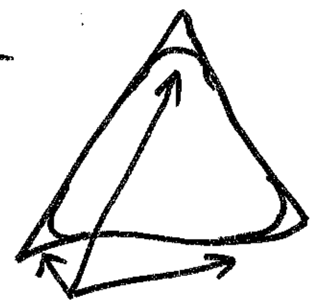
$$\iint_D K\,dA + \int_{\partial D} \kappa_g\,ds + \sum_k e_k = 2\pi$$



$$\iint_D K\,dA + \int_{\partial D} \kappa_g\,ds + \sum_k (\pi - i_k) = 2\pi$$

(c) Explain how the statement in part (b) may be derived from the statement in part (a).

Round out the corners and take limit as radius of curvature of rounding goes to zero.

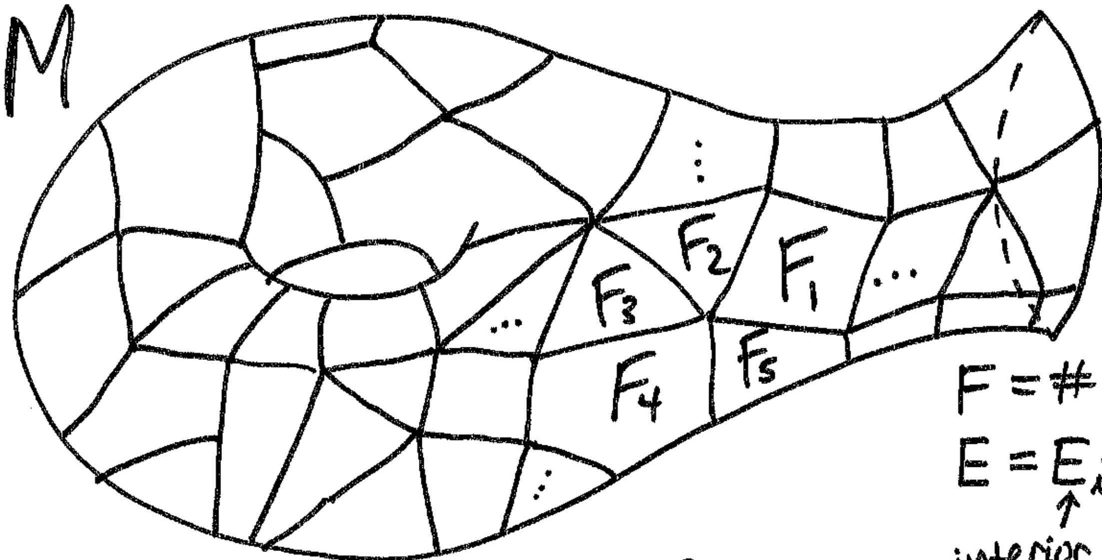


$$\lim \int \kappa_g\,ds = \lim \int \theta'(s)\,ds = \Delta\theta = e_k$$

at each corner.

(d) Using the statement in part (b), prove the Gauss Bonnet Theorem for a general compact surface with boundary.

M



∂M

$F = \#$ faces

$E = E_i + E_b$

E_i: interior edges; E_b: boundary edges

$V = V_i + V_b$

V_i: interior vertices; V_b: boundary vertices

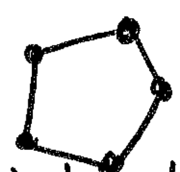
Add up over each face:

$$\iint_F K\,dA + \int_{\partial F} \kappa_g\,ds + \sum_{\text{corners}} (\pi - i) = 2\pi$$

$$\iint_M K\,dA + \int_{\partial M} \kappa_g\,ds \quad \boxed{\begin{matrix} +\,2\pi E_i \\ +\,\pi E_b \end{matrix}} \; \boxed{\begin{matrix} -\,2\pi V_i \\ -\,\pi V_b \end{matrix}} = 2\pi F$$

$$\boxed{+\pi E_b - \pi V_b} \leftarrow$$

may add in since $E_b = V_b$ on boundary:



which is always circles

$\therefore$

$$\iint_M K\,dA + \int_{\partial M} \kappa_g\,ds = 2\pi(F - E + V)$$
$$= 2\pi\,\chi(M).$$

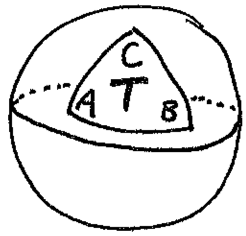
Problem 4. Applications of the Gauss Bonnet Theorem

For this problem, define a triangle in a surface M to be a region with three sides which are geodesics in M. Define the angles of the triangle to be the interior angles, just as you would normally do for a regular triangle in the plane.

(a) Prove that the area of a triangle on the standard unit sphere is given by the formula

$$\text{Area} = A + B + C - \pi$$

where A, B, C are the three angles of the triangle, measured in radians.



A triangle has Euler characteristic 1 and sides with $\kappa_g = 0$ on a sphere with $K = 1$. Thus, the Gauss Bonnet Theorem becomes

$$\iint_T K\,dA + \int_{\partial T} \kappa_g\,ds + \sum \text{exterior angles} = 2\pi \cdot \chi(T) \rightarrow$$

$$\text{Area} + 0 + (\pi - A) + (\pi - B) + (\pi - C) = 2\pi \longrightarrow$$

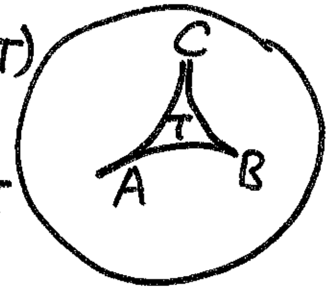
$$\text{Area} = A + B + C - \pi.$$

(b) Derive a similar formula for the area of a triangle in hyperbolic space where the Gauss curvature $K = -1$ everywhere.

$$\iint_T K\,dA + \int_{\partial T} \kappa_g\,ds + \sum \text{ext. angles} = 2\pi\chi(T)$$

$$-\text{Area} + 0 + (\pi - A) + (\pi - B) + (\pi - C) = 2\pi$$

$$\boxed{\text{Area} = \pi - A - B - C}.$$

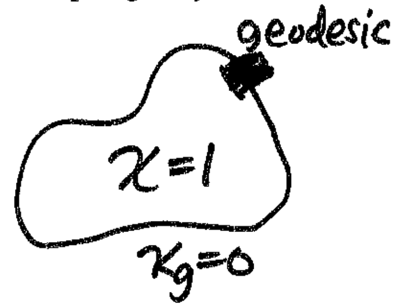


Poincare Disk

(c) Show that on a surface with $K \le 0$, no closed geodesic bounds a region topologically equivalent to a disk. If this existed,

$$0 \ge \iint_D K\,dA + \int_{\partial D} \kappa_g\,ds = 2\pi\chi = 2\pi$$

≤ 0 0 Gauss Bonnet



But $0 \ge 2\pi$ is false, so no such example exists.

Problem 5. For this problem we remind you that

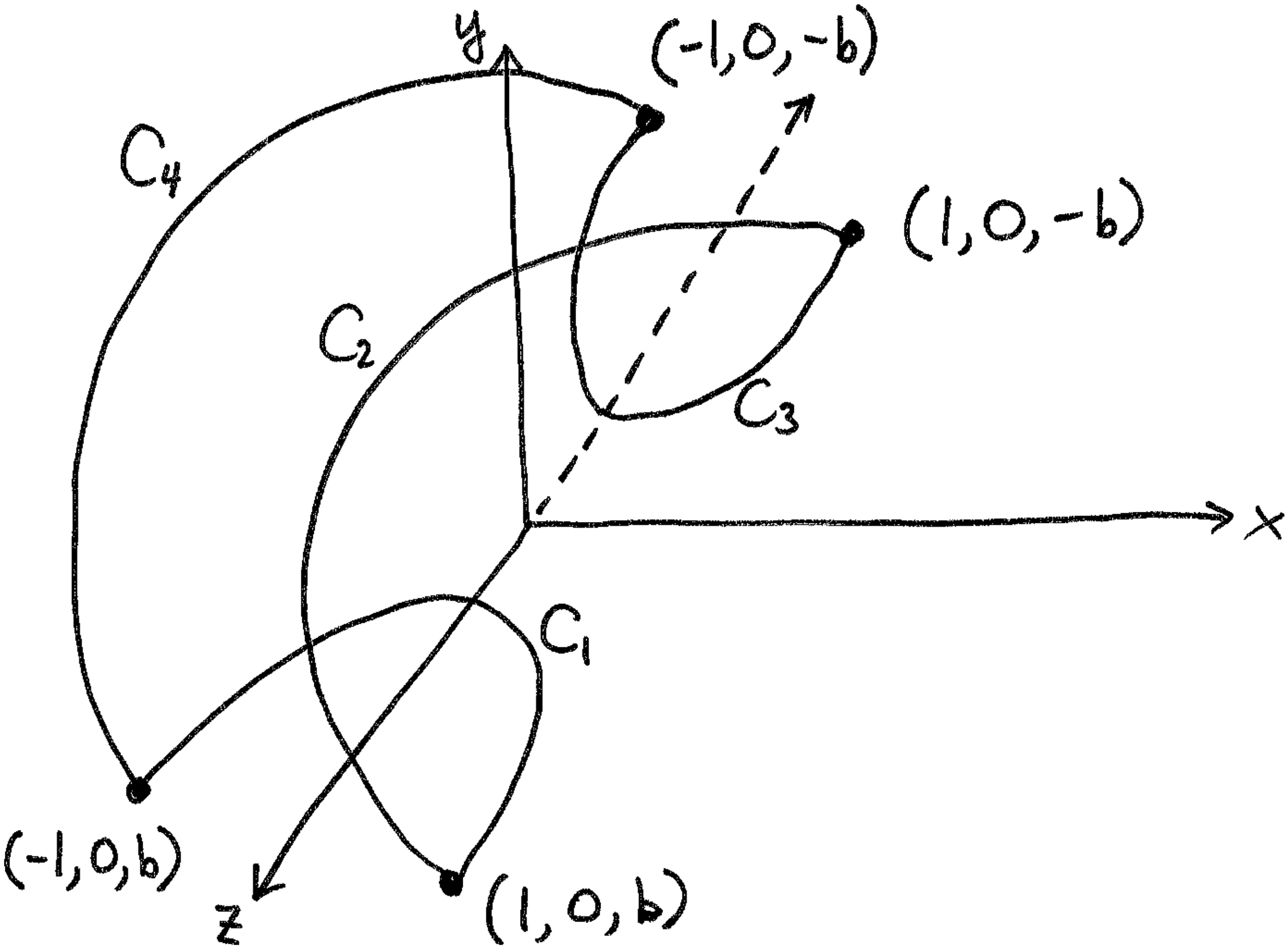
$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad 1+\sinh^2(t) = \cosh^2(t) = \frac{1}{2}(\cosh(2t)+1).$$

Define four curves as follows:

$$\begin{array}{lll} C_1: & \alpha_1(t) = (t, 0, \cosh(t)), & -1 \le t \le 1, \\ C_2: & \alpha_2(t) = (1, b\cos(t), b\sin(t)), & -\pi/2 \le t \le \pi/2, \\ C_3: & \alpha_3(t) = (t, 0, -\cosh(t)), & -1 \le t \le 1, \\ C_4: & \alpha_4(t) = (-1, b\cos(t), b\sin(t)), & -\pi/2 \le t \le \pi/2. \end{array}$$

where $b = \cosh(1)$.

(a) Sketch all four curves below. The quality of your sketch will help you later, so do a good job.



(b) Do the four curves, when joined together, form a closed loop?

Yes.

(c) Compute the curvature of C_2.

C_2 is: Half of a circle of radius b, so

$$\boxed{\kappa = \frac{1}{b} = \frac{1}{\cosh(1)}}.$$

(d) Compute the torsion of C_3.

C_3 is contained in the plane $y=0$,
so its torsion $\boxed{\tau = 0}$.

Problem 6. For this problem we remind you that

$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad 1+\sinh^2(t) = \cosh^2(t) = \frac{1}{2}(\cosh(2t)+1).$$

As in the previous problem, define four curves as follows:

$$\begin{array}{lll} C_1: & \alpha_1(t) = (t, 0, \cosh(t)), & -1 \le t \le 1, \\ C_2: & \alpha_2(t) = (1, b\cos(t), b\sin(t)), & -\pi/2 \le t \le \pi/2, \\ C_3: & \alpha_3(t) = (t, 0, -\cosh(t)), & -1 \le t \le 1, \\ C_4: & \alpha_4(t) = (-1, b\cos(t), b\sin(t)), & -\pi/2 \le t \le \pi/2. \end{array}$$

where $b = \cosh(1)$.

(a) Find a parametrized surface S with mean curvature $H = 0$ which has $C_1 \cup C_2 \cup C_3 \cup C_4$ as its boundary. Express S as the image of some $\vec{x}(u, v)$ for u, v in a certain range. (Hint: Being a good guesser is helpful for this problem.)

The catenoid:

$$\vec{x}(u,v) = (u, \cosh u \cdot \cos v, \cosh u \cdot \sin v),$$

$$-1 \le u \le 1$$

$$-\frac{\pi}{2} \le v \le \frac{\pi}{2}.$$

(b) Compute the area of S.

$$\text{Area} = \int_{-\pi/2}^{\pi/2} \int_{-1}^{1} |\vec{x}_u \times \vec{x}_v| \, du\, dv$$ = one way to do it...

$$= \int_{-1}^{1} \pi y \, ds$$ (revolve $y = \cosh x$ around half circle)

$$= \int_{-1}^{1} \pi \cosh x \sqrt{1 + \sinh^2 x} \cdot dx$$

$$= \int_{-1}^{1} \pi \cosh^2 x \, dx = \int_{-1}^{1} \frac{\pi}{2}(\cosh(2x) + 1)\, dx$$

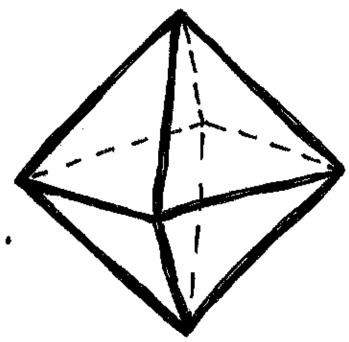
$$= \frac{\pi}{2}\left(\frac{1}{2}\sinh(2x) + x\right)\Big]_{-1}^{1} = \boxed{\frac{\pi}{2}\sinh 2 + \pi}.$$

Problem 7.

(a) State the definition of the Euler characteristic of a surface.

$$\chi = \underbrace{F}_{\#\text{faces}} - \underbrace{E}_{\#\text{edges}} + \underbrace{V}_{\#\text{vertices}}$$

of any triangulation of the surface.



(b) Using the triangulation of the sphere which looks like a regular octahedron (see figure), compute the Euler characteristic of the sphere. How many faces, edges, and vertices are there?

$F = 8$

$E = 12$

$V = 6$

$\chi = 8 - 12 + 6 = 2$.

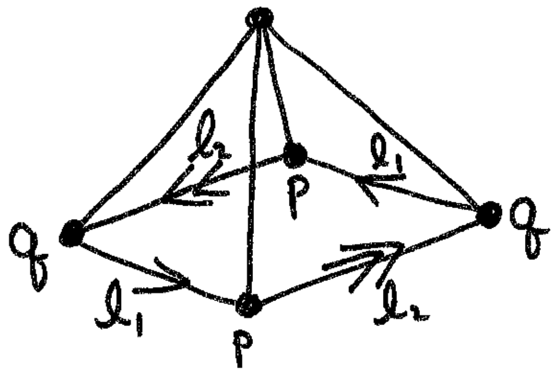
Let M be the surface obtained as follows: Take the top half (northern hemisphere) of a sphere which has a circle (the equator) as its boundary. Identify every point on the equator with its antipodal point exactly opposite it on the equator. The resulting abstractly defined surface, which we'll call M, does not have a boundary: An ant walking south on the 30° longitude line simply reappears on the "other side of the world" walking north on the 210° longitude line.

(c) Using the triangulation of the sphere from part (b), but with antipodal points identified, compute the Euler characteristic of M. How many faces, edges, and vertices are there in this triangulation of M?

$F = 4$
$E = 6$
$V = 3$

all exactly half of answer in part (b).

$\chi = 4 - 6 + 3 = 1$



(d) Deep application of Euler characteristic: Another way to view M (defined above) is as a quotient of the sphere where antipodal points of the sphere are identified as one point of M in a continuous way which results in a smooth surface. Does there exist a quotient of the sphere where three points are identified as one point in a continuous way which results in a smooth surface? Prove it.

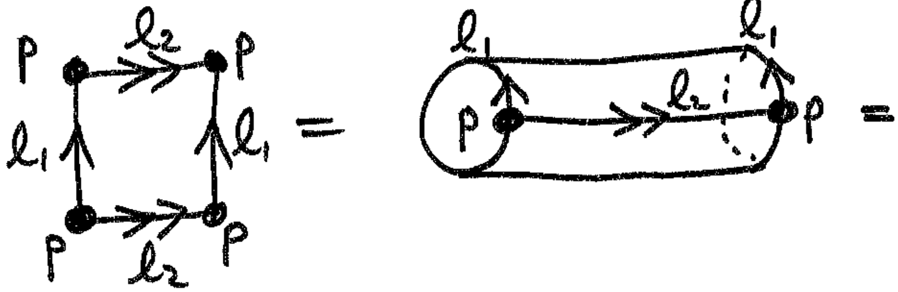
In part (c), we saw that identifying 2 points gave $\chi(M) = \chi(\text{Sphere})/2 = 2/2 = 1$.

Similarly, identifying 3 points would imply the new smooth surface had $\chi = \chi(\text{Sphere})/3 = 2/3$.

But Euler characteristics must be integers, so this is not possible.

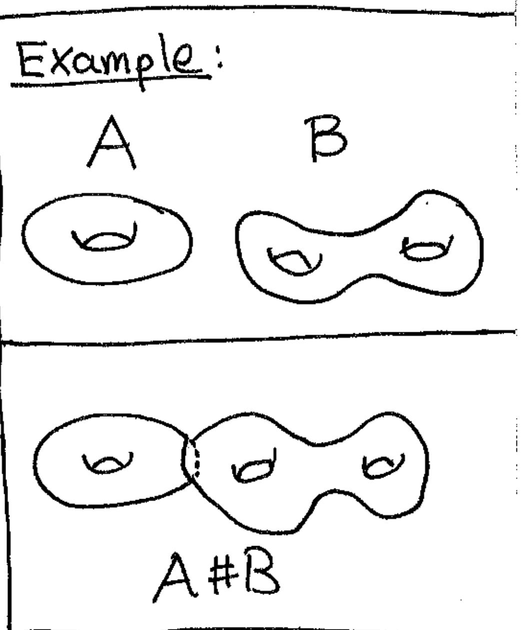
Problem 8.

(a) Compute the Euler characteristic of a torus (surface of a donut).



= Stick end together to get donut / torus.

$$\left.\begin{array}{l} F=1 \\ E=2 \\ V=1 \end{array}\right\} \chi = 1-2+1 = 0.$$



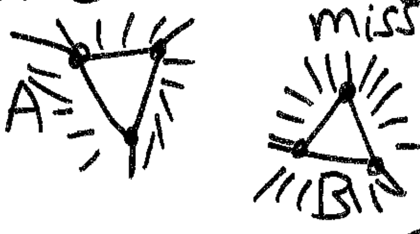
(b) Prove the following connect sum identity for Euler characteristics,

$$\chi(A\#B) = \chi(A) + \chi(B) - 2$$

where A and B are compact surfaces with boundary.

$-1-1=-2$: Remove 2 faces one from A, one from B.

Glue surface together along missing faces.



$$\left.\begin{array}{l} F \text{ goes down by } 2 \\ E \text{ goes down by } 3 \\ V \text{ goes down by } 3 \end{array}\right\} \begin{array}{l} \chi(A\#B) = \\ \chi(A)+\chi(B)-2. \end{array}$$

cancel out

(c) Use parts (a) and (b) to compute the Euler characteristic of a surface of genus g (the surface of a donut with g holes in it). Prove that your answer is correct.

Example: Surface of genus 4.

Each time we "connect sum" with a torus, we get 1 more hole and Euler characteristic goes down by 2:

$$\chi(A\#T) = \chi(A) + \underset{0.}{\chi(T)} - 2$$
$$= \chi(A) - 2$$

Since 0 holes $\rightarrow \chi = 2$

g holes $\rightarrow \boxed{\chi = 2-2g}$.

Problem 9. (The Hyperbolic Plane / Poincare Disk)

Define H to be the disk of radius 2 in the plane minus the bounding circle. Define

$$U \circ V = \frac{U \cdot V}{(1 - r^2/4)^2}$$

where U and V are vectors based at the same point p which is a distance r from the origin according to the normal Euclidean distance.

(a) What is the length of the vector based at $(1,1)$ with coordinates $(1,5)$ (so you might draw it as going from $(1,1)$ to $(2,6)$, even though of course it is just in the tangent plane at $(1,1)$).

Since $r=\sqrt{2}$, $|V|^2 = V \circ V = \dfrac{V \cdot V}{(1-\frac{1}{2})^2} = \dfrac{(1,5)\cdot(1,5)}{\frac{1}{4}} = 4(26) = 104 \rightarrow \boxed{|V| = 2\sqrt{26}}$.

(b) What is the length of the circle $r = 1$ according to the $\circ$ metric?

Lengths change by a factor of $\sqrt{\dfrac{1}{(1-r^2/4)^2}}$ as in part (a).

Thus,

$$l = 2\pi \cdot 1 \cdot \sqrt{\frac{1}{(1-1/4)^2}} = \frac{2\pi}{(1-\frac{1}{4})} = \frac{8\pi}{3}.$$

(c) What is the area of the disk $r \leq 1$ according to the $\circ$ metric? (You haven't worked this problem before, so think carefully.)

Areas change by a factor of $\dfrac{1}{(1-\frac{r^2}{4})^2}$ (since $A = h \cdot w$).

Thus

$$\text{Area} = \int_0^1 \left[2\pi r \cdot dr\right] \frac{1}{(1-\frac{r^2}{4})^2}$$

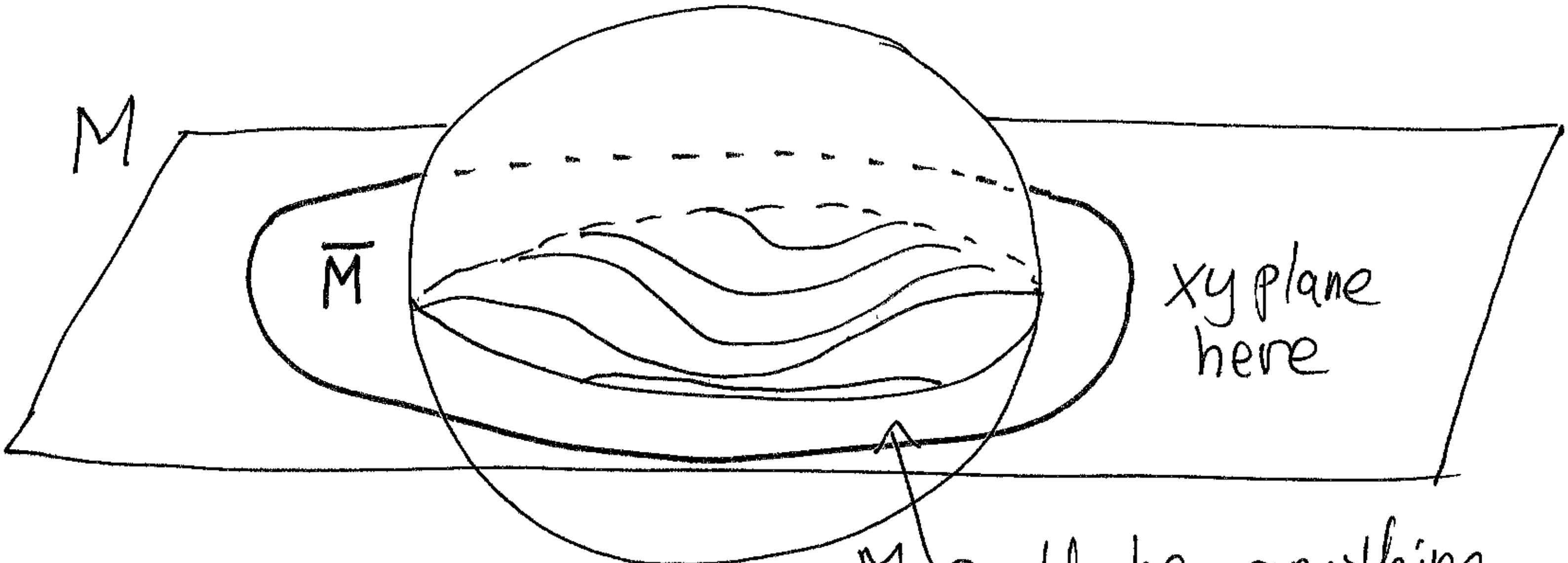
$$= \int_0^1 \frac{2\pi r}{(1-\frac{r^2}{4})^2}\, dr = \left.\frac{4\pi}{1-\frac{r^2}{4}}\right]_0^1$$

$$= \frac{4\pi}{(\frac{3}{4})} - 4\pi = \frac{4\pi}{3}.$$

Problem 10. (The Rigidity of the Plane)

Suppose M is a surface without boundary which is precisely equal to the xy-plane outside the ball of radius 100. Prove that if M has Gauss curvature $K \geq 0$ everywhere, then M has $K = 0$ *everywhere*.

(You may assume that M has the same topology as the xy-plane, if you like.)



M could be anything inside the ball of radius 100, but must have $K \geqslant 0$.

Let $\bar{M}$ be M inside the ball of radius 150. $\bar{M}$ is hence a disk with $\chi = 1$.

Also, $\partial\bar{M}$ is a circle of radius 150, so $\kappa = \frac{1}{150}$ and the circumference is $2\pi \cdot 150$. Hence, by Gauss-Bonnet

$$\iint_{\bar{M}} K\,dA + \int_{\partial\bar{M}} \kappa_g\,ds = 2\pi\cdot\chi \longrightarrow$$

$$\iint_{\bar{M}} K\,dA + 2\pi\cdot 150\cdot\frac{1}{150} = 2\pi\cdot 1 \longrightarrow$$

$$\boxed{\iint_{\bar{M}} K\,dA = 0.}$$

since $K>0$ anywhere would make positive.

Hence, $K \geqq 0$ implies $K = 0$ everywhere