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Midterm, Math 421
Differential Geometry: Curves and Surfaces in $\mathbf{R}^3$

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Your Name: Key

Honor Pledge Signature:

Instructions: This is a 75 minute, closed book exam. You may bring one $8\frac{1}{2}'' \times 11''$ piece of paper with anything you like written on it to use during the exam, but nothing else. No collaboration on this exam is allowed. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points. Good luck on the exam!

Problem 1. Consider the curve in $\mathbf{R}^3$ parametrized by

$$\alpha(t) = (\cos(5t), \sin(5t), 12t).$$

(a) What is the speed of α? Find a *unit speed* reparametrization $\beta(s)$.

$$v = |\alpha'(t)| = |(-5\sin 5t, 5\cos 5t, 12)| = \sqrt{5^2+12^2} = \boxed{13}.$$

$$s = 13t \ ; \ t = \frac{s}{13}$$

$$\beta(s) = \alpha(t) = \boxed{\left(\cos\frac{5s}{13}, \sin\frac{5s}{13}, \frac{12s}{13}\right)}$$

(b) Using the unit speed reparametrization $\beta(s)$, compute the curvature κ of the curve.

$$\beta'(s) = \left(-\frac{5}{13}\sin\frac{5s}{13}, \frac{5}{13}\cos\frac{5s}{13}, \frac{12}{13}\right) = T$$

$$\beta''(s) = \left(-\frac{25}{169}\cos\frac{5s}{13}, -\frac{25}{169}\sin\frac{5s}{13}, 0\right) = \kappa N$$

$$\kappa = |\beta''(s)| = \boxed{\frac{25}{169}}$$

(c) Compute all three vectors of the Frenet frame (T, N, B) for the curve $\beta(s)$.

$$T = \beta'(s) = \left(-\frac{5}{13}\sin\frac{5s}{13}, \frac{5}{13}\cos\frac{5s}{13}, \frac{12}{13}\right)$$

$$N = \frac{\beta''(s)}{\kappa} = \left(-\cos\frac{5s}{13}, -\sin\frac{5s}{13}, 0\right)$$

$$B = T \times N = \left(\frac{12}{13}\sin\frac{5s}{13}, -\frac{12}{13}\cos\frac{5s}{13}, \frac{5}{13}\right)$$

(d) Compute the torsion τ of the curve $\beta(s)$.

$$\tau = -B'(s)\cdot N \qquad (\text{or } N'(s)\cdot B)$$

$$= -\left(\frac{60}{169}\cos\frac{5s}{13}, \frac{60}{169}\sin\frac{5s}{13}, 0\right)\cdot N$$

$$= \boxed{\frac{60}{169}}$$

Problem 2.

(a) Prove that a curve in the xy-plane with constant curvature $\kappa = 1/R$ is a circle of radius R.

Let $\beta(s)$ be its constant-speed parametrization.

$$\frac{d}{ds}\left(\beta(s) + R\vec{N}\right) = \beta'(s) + RN'(s)$$

$$= T + R(-\kappa T + \tau B)$$

$\tau = 0$ since curve is in plane

$$= T + R\left(-\tfrac{1}{R}T\right) = 0.$$

Hence, $\beta(s) + R\vec{N} = p \to |\beta(s) - p| = R$, a circle.

(b) Prove that a surface with zero shape operator is contained in a plane.

Let $f(t) = (q - \alpha(t)) \cdot U(\alpha(t))$, where $\boxed{\begin{array}{l}\alpha(0) = q \\ \alpha(1) = p\end{array}}$

for any $q \in M$ and some fixed $p \in M$. Then since

$$f'(t) = -\alpha'(t) \cdot U(\alpha(t)) + (q - \alpha(t)) \nabla_{\alpha'(t)} U$$

$$= 0 + 0 = 0,$$

and since $f(0) = 0$, it follows that $f(t) = 0$, $\forall t$. Let $t=1$:

$$(q - p) \cdot U(p) = 0, \quad \forall q \in M \longrightarrow \text{plane}.$$

(c) Prove that a surface with principle curvatures $k_1 = k_2 = 1/R$ is contained in a sphere of radius R.

$$\frac{\partial}{\partial u}\left(\vec{X}(u,v) + R\vec{U}\right) = \vec{X}_u + R\nabla_{X_u}\vec{U}$$

$$= \vec{X}_u - RS_p(\vec{X}_u) = \vec{X}_u - R\left(\tfrac{1}{R}\vec{X}_u\right) = 0$$

since $S_p(v) = \frac{1}{R}v$ for all v. Similarly, $\frac{\partial}{\partial v}(\ ) = 0$.

Hence, $\vec{X}(u,v) + R\vec{U} = \text{constant} = p \Rightarrow$

$$|\vec{X}(u,v) - \vec{p}| = R, \quad \text{a sphere.}$$

Problem 3.

(a) Given a surface M with unit normal vector field U, define the shape operator $S_p(v)$ at the point p on M, where v is a tangent vector to M at p.

$$S_p(v) = -\nabla_v U$$

(b) Prove that $S_p(v)$ is also a tangent vector to M at p. (Hence, $S_p : T_pM \to T_pM$.)

$$0 = v[1] = v[U \cdot U] = 2U \cdot \nabla_v U \Rightarrow$$

$$\nabla_v U \perp U \Rightarrow -\nabla_v U \in T_pM$$

Hence, $S_p(v)$ is also a tangent vector.

(c) Prove that S_p is symmetric as follows: Given a coordinate chart $\vec{x}(u,v)$, show that

$$S_p(\vec{x}_u) \cdot \vec{x}_v = \vec{x}_u \cdot S_p(\vec{x}_v).$$

$$0 = \vec{U} \cdot \vec{X}_v$$

$$0 = \vec{X}_u[\vec{U} \cdot \vec{X}_v] = (\nabla_{\vec{X}_u} \vec{U}) \cdot \vec{X}_v + \vec{U} \cdot \vec{X}_{uv}$$

$$\therefore\ S_p(\vec{X}_u) \cdot \vec{X}_v = \vec{U} \cdot \vec{X}_{uv} = S_p(\vec{X}_v) \cdot \vec{X}_u$$

Thus, S_p is symmetric $(S_p(v) \cdot w = v \cdot S_p(w),\ \forall v, w)$ since it is symmetric on a set of basis vectors $\{\vec{X}_u, \vec{X}_v\}$.

Problem 4. Consider a smooth surface M in $\mathbf{R}^3$. You may use theorems from class (without reproving them here) if that helps with parts of this problem.

(a) Let k_1 and k_2 be the principle curvatures (the eigenvalues of the shape operator S_p) at each point of M. Express the mean curvature H and the Gauss curvature K in terms of k_1 and k_2.

$$H = \frac{k_1 + k_2}{2}$$

$$K = k_1 k_2$$

(b) Prove that $H^2 \geq K$ at each point. Hint: $(a+b)^2 = 4ab + (a-b)^2$.

$$\left(\frac{k_1+k_2}{2}\right)^2 = k_1 k_2 + \left(\frac{k_1 - k_2}{2}\right)^2$$

$$H^2 = K + \left(\frac{k_1-k_2}{2}\right)^2 \geq K$$

(c) Prove that if $\int_M (H^2 - K)\, dA = 0$, then M is contained in either a plane or a sphere.

$$0 = \int_M (H^2 - K)\, dA = \int_M \left(\frac{k_1-k_2}{2}\right)^2 dA \Rightarrow k_1 = k_2$$

everywhere

$\Downarrow$

M is contained in either a plane or a sphere. $\Leftarrow$ M is umbilic

(d) Prove that there does not exist a compact minimal surface M in $\mathbf{R}^3$. You may assume M is a smooth surface without boundary contained inside a large sphere.

We proved a theorem that all compact surfaces have a point with $K > 0$. At that point, then

$$H^2 \geq K > 0.$$

But $H = 0$ everywhere, so a compact minimal surface is not possible.

Problem 5. A ruled surface has a parametization, or ruling, $x(u, v) = \beta(u) + v\delta(u)$. A doubly ruled surface is a surface with two different rulings. A triply ruled surface is a surface with three different rulings. In the above defintions, different rulings are required to have different directions for $\delta(u)$ at every point.

(a) Prove that for every point p on a ruled surface, there must be a tangent direction $\vec{u}$ such that the normal curvature $k(\vec{u}) = 0$.

Let $\vec{u} = \delta(u)$. Then cross section of surface with plane spanned by $\vec{U}$ and $\vec{u}$ is a straight line, so

$$k(\vec{u}) = \pm \kappa = 0.$$

(b) Prove that at every point on a ruled surface, the principle curvatures k_1 and k_2 must have $k_1 \geq 0$ and $k_2 \leq 0$.

$$\left.\begin{aligned} k_1 &= \max_{\vec{u}} k(\vec{u}) \geq 0 \\ k_2 &= \min_{\vec{u}} k(\vec{u}) \leq 0 \end{aligned}\right\} \text{since } k(\vec{u}) = 0 \text{ as above, when } \vec{u} = \delta(u).$$

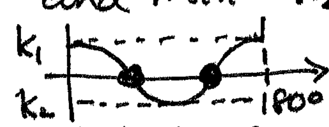
(c) Prove that at every point of a surface that is *doubly* ruled (but not triply ruled), $k_1 > 0$ and $k_2 < 0$. Note the strict inequality - you must prove that neither can be zero.

Let $\vec{u} = \cos\theta \vec{V}_1 + \sin\theta \vec{V}_2$, where V_1, V_2 are principle curvature directions. Then

$$k(\vec{u}) = \vec{u} \cdot S_p(\vec{u}) = k_1 \cos^2\theta + k_2 \sin^2\theta = k(\theta)$$

$$= k_1\left(\frac{1+\cos 2\theta}{2}\right) + k_2\left(\frac{1-\cos 2\theta}{2}\right) = \text{sine wave with max} = k_1 \text{ and min} = k_2$$

$\boxed{\text{The only way to get exactly two zeros is for } k_1 > 0 \text{ and } k_2 < 0.}$



(d) Prove that at every point of a *triply* ruled surface, $k_1 = 0$ and $k_2 = 0$, forcing the surface to be contained in a plane.

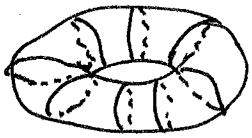
The only way for $k(\theta) =$ to have three or more zeros is for $k_1 = 0$ and $k_2 = 0$ everywhere.

Problem 6. (A Surface in 4-Dimensional Euclidean Space)

Consider the surface in 4-dimensional Euclidean space parametrized by

$$\vec{x}(u,v) = (3\cos(u), 3\sin(u), 4\cos(v), 4\sin(v)).$$

(a) If we allow $-\infty \leq u, v \leq \infty$, this parametrization wraps around the surface infinity many times. Topologically, what is this surface? (Possible answers: a plane, a disk, a sphere, a torus (surface of a donut), the surface of a donut with two holes, etc.)



torus, since this is a "circle of circles".

(b) Compute $\vec{x}_u$ and $\vec{x}_v$.

$$\vec{X}_u = (-3\sin u, 3\cos u, 0, 0)$$
$$\vec{X}_v = (0, 0, -4\sin v, 4\cos v)$$

(c) Since this surface is not embedded in 3-dimensions, our usual unit normal vector $\vec{U}$ is not well defined, meaning that our usual shape operator is not well defined. Hence, we cannot compute l, m, or n. However, we can compute E, F, and G. What are they?

$$E = \vec{X}_u \cdot \vec{X}_u = 9$$
$$G = \vec{X}_v \cdot \vec{X}_v = 16 \quad ; \quad F = X_u \cdot X_v = 0$$

(d) Since the shape operator is not well-defined, the principal curvatures, principal curvature directions, and mean curvature of this surface are not well defined in the usual sense. However, we do have a formula for the Gauss curvature of this surfaces in terms of E, F, G when $F = 0$, which is

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right]$$

Compute K for this surface.

$K = 0$ since $E_v = 0$ and $G_u = 0$.

(e) Prove that this surface with this metric (meaning this E, F, G) cannot be embedded as a smooth compact surface in 3 dimensional Euclidean space.

Every smooth compact surface embedded in $\mathbb{R}^3$ has at least one point with $K > 0$. Hence, this surface with this metric, which has $K = 0$, is not embeddable in $\mathbb{R}^3$.