Final Exam, Math 421
Differential Geometry: Curves and Surfaces in $\mathbf{R}^3$

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Thursday, May 5, 2016

Your Name: Key

Honor Pledge Signature:

Instructions: This is a 3 hour, closed book exam. You may bring one $8\frac{1}{2}'' \times 11''$ piece of paper with anything you like written on it to use during the exam, but nothing else. No collaboration on this exam is allowed. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points. Good luck on the exam!

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
10	12	
Total	120	

Problem 1. Consider the curve parametrized by

$$\alpha(t) = (t^3 + t, 2t^3 + 2t,$$ [redacted] $$. \; 3t^3 + 3t)$$

(a) What is the speed of $\alpha(t)$?

$$v = |\alpha'(t)| = |(1,2,3)\cdot(3t^2+1)|$$

$$= \sqrt{14}\cdot(3t^2+1)$$

(b) What is the curvature of $\alpha(t)$?

$\alpha(t) = (t^3+t)(1,2,3)$ parametrizes a line, so $\kappa = 0$.

(c) Find a *unit speed* reparametrization $\beta(s)$ of this curve.

$$\beta(s) = \frac{s}{\sqrt{14}}\cdot(1,2,3)$$

(d) Define a surface on which $\beta(s)$ is a geodesic.

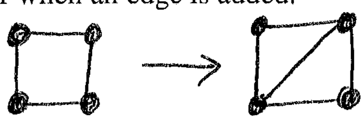
Any plane which contains $\beta(s)$, for example $2x - y = 0$.

Problem 2.

(a) Define the Euler characteristic of a surface.

$$\chi = F - E + V$$

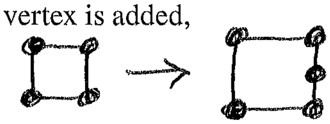
(b) Prove that the Euler characteristic of a triangulation does not change when a vertex is added, or when an edge is added.



$E \to E+1$

$F \to F+1$

χ is unchanged.



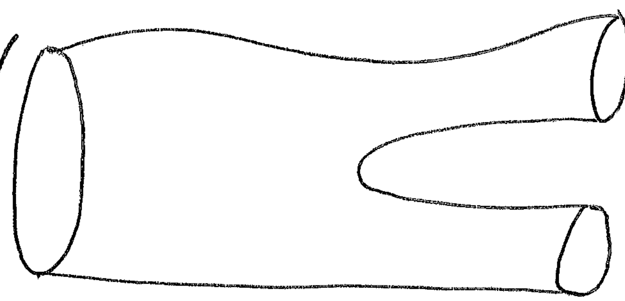
$E \to E+1$

$V \to V+1$

χ is unchanged.

(c) What is the Euler characteristic of this surface?

$\chi\Big($



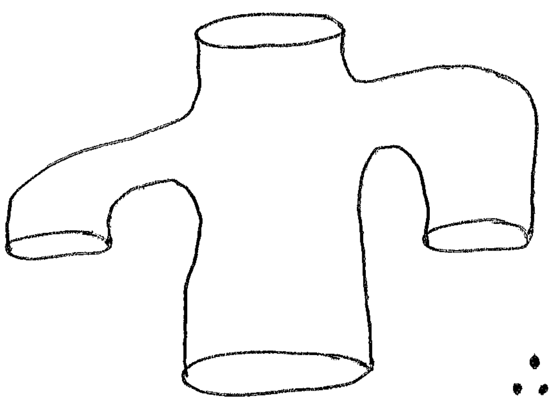
$\Big)$

$\chi(\text{sphere}) = 2$, so

$= 2-1-1-1 = -1$

since there are 3 missing faces.

(d) What is the average value of the Gauss curvature of this surface of total area 10? You may assume that the boundary curves are geodesics.



$$\iint K\,dA + \int \kappa_g\,ds = 2\pi\chi$$

$\int \kappa_g\,ds$: zero

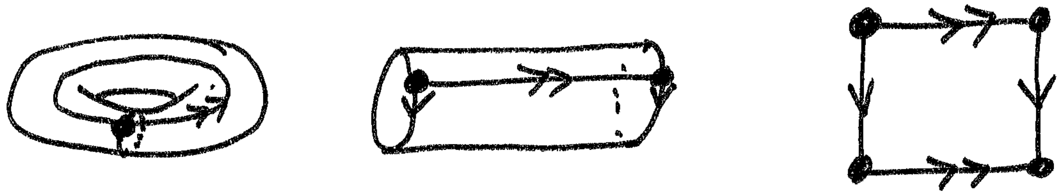
χ: $2-1-1-1-1=-2$

$$\iint K\,dA = -4\pi$$

$$\therefore \bar{K} = \frac{1}{10}\iint K\,dA = -\frac{4\pi}{10} = -\frac{2\pi}{5}$$

Problem 3.

(a) Compute the Euler characteristic of a torus (surface of a donut).



$$\left.\begin{array}{l} V=1 \\ E=2 \\ F=1 \end{array}\right\} \chi = 1-2+1 = 0.$$

(b) Prove the following connect sum identity for Euler characteristics,

$$\chi(A\#B) = \chi(A) + \chi(B) - 2$$

where A and B are compact surfaces with boundary.

Given triangulations on

remove 1 face   remove 1 face (and equal numbers of vertices and edges)

cancels out

to get a triangulation on

with 2 fewer faces than before.

(c) Use parts (a) and (b) to compute the Euler characteristic of a surface of genus g (the surface of a donut with g holes in it). Prove that your answer is correct.

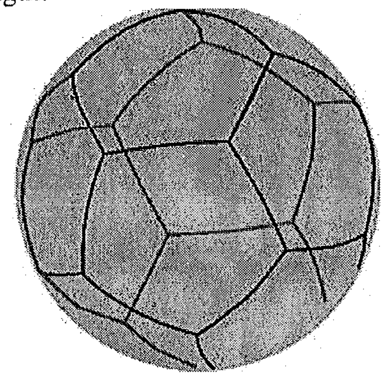
$\chi($  $)$ (g holes)

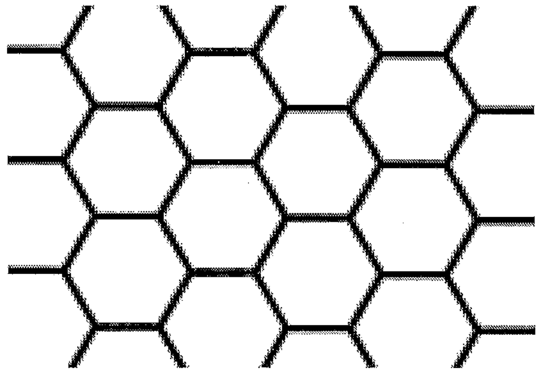
$= \chi($  $)$ (g tori)

$= 0 - 2 - 2 - 2$ (g-1 times)

$= -2(g-1) = 2-2g.$

Problem 4. The image on the left is a regular pentagonal tiling of the unit sphere (analogous to the image on the right, where flat space is tiled by hexagons.) To be clear, all of the angles in the left image are the same and all of the sides (each of which are geodesics) have the same length.





Given that the unit sphere has Gauss curvature $K = 1$, what is the area of each of the pentagons?

(You may not use the fact that an icosahedron has 12 sides, without proof, nor do we recommend that you pursue that direction. You may, however, use some version of the Gauss-Bonnet theorem.)

$$\underbrace{\iint K\,dA}_{\text{face of pentagon}} + \underbrace{\int \kappa_g}_{\text{edges of pentagon}} + \sum_k \left(\pi - \frac{2\pi}{3}\right) = 2\pi$$

interior angles are all $120° = 2\pi/3$.

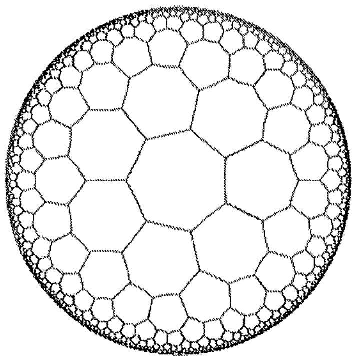
$$K \cdot A + 0 + 5 \cdot \frac{\pi}{3} = 2\pi$$

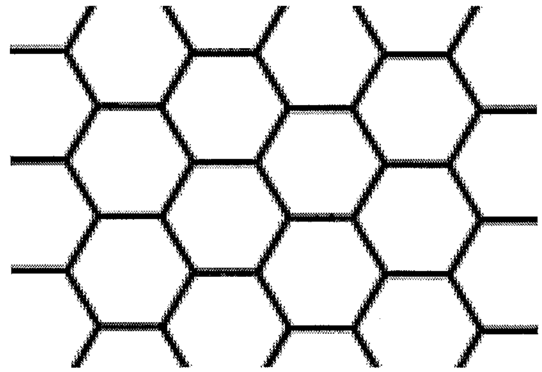
$$K \cdot A = \frac{\pi}{3}$$

$$A = \frac{\pi}{3}$$

(since $K = 1$)

Problem 5. The image on the left is a regular heptagonal tiling of hyperbolic space (analogous to the image on the right, where flat space is tiled by hexagons.) To be clear, all of the angles (as measured in hyperbolic space) in the left image are the same and all of the sides (each of which are geodesics) have the same length. (Geodesics are straight lines in the Klein model of hyperbolic space, depicted below.)





Given that hyperbolic space has Gauss curvature $K = -1$, what is the area of each of the heptagons?

Again, since all the angles are equal, and 3 angles add up to 2π, the interior angles are all $2\pi/3$.

$$\underbrace{\iint K\,dA}_{\text{face of a heptagon}} + \underbrace{\int \kappa_g}_{\text{edges of heptagon}} + \sum_k \left(\pi - \frac{2\pi}{3}\right) = 2\pi$$

$$K \cdot A + 0 + 7 \cdot \frac{\pi}{3} = 2\pi$$

$$KA = -\frac{\pi}{3}$$

$$A = \frac{\pi}{3}$$

(since $K = -1$)

Problem 6. Define a geodesic of a surface M to be any curve $\alpha(t)$ on M such that $\alpha''(t)$ is perpendicular to M.

(a) Prove that a geodesic has constant speed.

$$\frac{d}{dt} v^2 = \frac{d}{dt} \alpha'(t) \cdot \alpha'(t) = 2\alpha'(t) \cdot \alpha''(t) = 0$$

since $\alpha'' \perp M$ and hence $\alpha'' \perp \alpha'$.

(b) Give the definition of geodesic curvature for a general curve on a surface.

$$\boxed{v^2 \kappa_g = \alpha''(t) \cdot J(T)}$$

(rotation of T $= U \times T$)

speed of curve $= |\alpha'(t)|$

(c) Prove that a geodesic has zero geodesic curvature.

$\alpha''(t) \perp M$, so $\alpha''(t) \cdot J(T) = 0$.

Hence, $\kappa_g = 0$.

(d) Suppose α is a geodesic on the standard unit sphere. Prove that it is contained in a plane.

By symmetry, the geodesic curvature of a great circle (like the equator) is zero. Hence, all great circles are geodesics. Since the initial position and initial direction determines a geodesic, and since all initial positions and directions correspond to some great circle, all geodesics are great circles, which by definition are contained in a plane going through the origin.

Problem 7. Suppose α is a geodesic on M and is also contained in a plane P. Prove that α is also a line of curvature of M. (Recall that a line of curvature is any curve whose tangent direction T is an eigenvector of the shape operator at every point.)

Without loss of generality, we suppose $\alpha(s)$ is a unit speed geodesic. Then

$$\alpha''(s) = T'(s) = \kappa N \perp M \iff N = \pm U.$$

Let's choose $U = N$.

Since $\alpha(s)$ is in a plane, $\tau = 0$. Hence,

$$N'(s) = -\kappa T + \tau B$$

$$U'(s) = -\kappa T$$

$$\nabla_T U = -\kappa T$$

$$S_p(T) = \kappa T$$

Hence, T is an eigenvector of the shape operator everywhere, so $\alpha(s)$ is a line of curvature of M.

Problem 8. Given a coordinate chart with $F = 0$, define $\vec{\mathcal{E}}_1 = \vec{x}_u/\sqrt{E}$ and $\vec{\mathcal{E}}_2 = \vec{x}_v/\sqrt{G}$ as an orthonormal frame on the surface, as usual. Let $\beta(s)$ be any unit speed curve. Using the definitions of the quantities below, prove that

$$\kappa_g = \frac{d\theta}{ds} + \omega_{21}(\beta'(s)),$$

where $\theta(s)$ is the angle that $\beta'(s)$ makes with $\vec{\mathcal{E}}_1$ (in the direction of $\vec{\mathcal{E}}_2$).

$$T = \beta'(s) = \cos\theta(s)\cdot\vec{\mathcal{E}}_1 + \sin\theta(s)\cdot\vec{\mathcal{E}}_2 \quad \underline{\text{by def.}}$$

$$\beta''(s) = \cos\theta(s)\cdot\nabla_{\beta'(s)}\vec{\mathcal{E}}_1 + \sin\theta(s)\cdot\nabla_{\beta'(s)}\vec{\mathcal{E}}_2$$
$$-\sin\theta(s)\cdot\theta'(s)\vec{\mathcal{E}}_1 + \cos\theta(s)\cdot\theta'(s)\vec{\mathcal{E}}_2$$

Substitute $\nabla_{\beta'(s)}\vec{\mathcal{E}}_1 = \omega_{21}(\beta'(s))\vec{\mathcal{E}}_2$

$\nabla_{\beta'(s)}\vec{\mathcal{E}}_2 = -\omega_{21}(\beta'(s))\vec{\mathcal{E}}_1$ to get

$$\beta''(s) = (\theta'(s) + \omega_{21}(\beta'(s)))(-\sin\theta(s)\vec{\mathcal{E}}_1 + \cos\theta(s)\vec{\mathcal{E}}_2)$$
$$= \kappa_g\cdot(\mathcal{U}\times T) \quad \text{by def. of } \kappa_g \text{ too.}$$

But $\mathcal{U}\times T = \mathcal{U}\times(\cos\theta(s)\vec{\mathcal{E}}_1 + \sin\theta(s)\vec{\mathcal{E}}_2)$

$= \cos\theta(s)(\vec{\mathcal{E}}_2) + \sin\theta(s)(-\vec{\mathcal{E}}_1) =$

since $\vec{\mathcal{U}} = \vec{\mathcal{E}}_1\times\vec{\mathcal{E}}_2$.

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Hence, $\boxed{\kappa_g = \theta'(s) + \omega_{21}(\beta'(s))}$.

Problem 9. Prove the Gauss Bonnet Theorem for a Disk using the identity from problem 8. Show all steps.

$\beta(s) = \vec{X}(u(s), v(s))$

$\beta'(s) = u'(s) \cdot \vec{X}_u + v'(s)\vec{X}_v$

$$\omega_{21}(\beta'(s)) = \left(\nabla_{\beta'(s)} \vec{\mathcal{E}}_1\right) \cdot \vec{\mathcal{E}}_2 \quad ;$$

$$= \left(u'(s)\left(\frac{\vec{X}_u}{\sqrt{E}}\right)_u + v'(s)\left(\frac{\vec{X}_u}{\sqrt{E}}\right)_v \right) \cdot \frac{\vec{X}_v}{\sqrt{G}}$$

$$= u'(s)\,\frac{\vec{X}_{uu} \cdot \vec{X}_v}{\sqrt{EG}} + v'(s)\,\frac{\vec{X}_{uv} \cdot \vec{X}_v}{\sqrt{EG}}$$

$$= u'(s)\left(-\frac{E_v}{2\sqrt{EG}}\right) + v'(s)\left(\frac{G_u}{2\sqrt{EG}}\right)$$

$0 = F = X_u \cdot X_v$

$0 = F_u = X_{uu} \cdot X_v + X_u \cdot X_{uv}$

$X_{uu} \cdot X_v = -X_u \cdot X_{uv}$

$= -\frac{1}{2}(X_u \cdot X_u)_v$

$= -\frac{E_v}{2}$

$X_{uv} \cdot X_v = \frac{1}{2}(X_v \cdot X_v)_u$

$= \frac{G_u}{2}$

Thus, $2\pi - \int_{\partial D} \kappa_g\, ds =$

$$= \int_{\partial D}\left(\frac{d\theta}{ds} - \kappa_g\right) ds = \int_{\partial D} -\omega_{21}(\beta'(s))\, ds$$

$$= \int_{\partial D}\left[\frac{du}{ds}\left(\frac{E_v}{2\sqrt{EG}}\right) - \frac{dv}{ds}\left(\frac{G_u}{2\sqrt{EG}}\right)\right] ds$$

$$= \int_{\partial D} \frac{E_v}{2\sqrt{EG}} \cdot du - \frac{G_u}{2\sqrt{EG}}\, dv$$

$$= \int_D \left[-\left(\frac{E_v}{2\sqrt{EG}}\right)_v - \left(\frac{G_u}{2\sqrt{EG}}\right)_u\right] du\,dv$$

$$= \int_D \underbrace{-\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right]}_{K} \cdot \underbrace{\sqrt{EG}\, du\,dv}_{dA}$$

$$= \int_D K \cdot dA = \int_D K \cdot dA.$$

Problem 10. For this problem, you may use the first variation of area formula

$$A'(0) = -\int_M 2H\,(\vec{U}\cdot\vec{V})\,dA,$$

where $A(s)$ is the area of the surface $M(s)$ which, at $s=0$, has unit normal vector $\vec{U}$ and mean curvature H, and is flowing with velocity $\vec{V}$ as s increases. (The above formula is true as long as $\vec{V}=0$ on the boundary of M, which is all you will need for this problem.)

(a) Prove that a soap bubble, which minimizes area A among all surfaces which enclose a given, fixed volume V_0 of air, must have constant mean curvature everywhere.

$0 = V'(0) = \int_M \vec{U}\cdot\vec{V}\,dA$ is required to preserve volume.

Let $\eta = \vec{U}\cdot\vec{V}$. Then ~~we~~ a soap bubble has

$0 = A'(0) = -\int_M 2H\cdot\eta\,dA$ (for all η with $\int_M \eta\,dA = 0$)

Choose $\eta = H - \bar{H}$, where $\bar{H} = \frac{1}{A}\int_M H\,dA$. Then

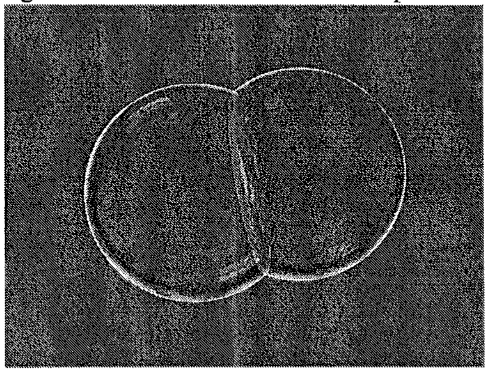
$\int (H-\bar{H})\,dA = 0$

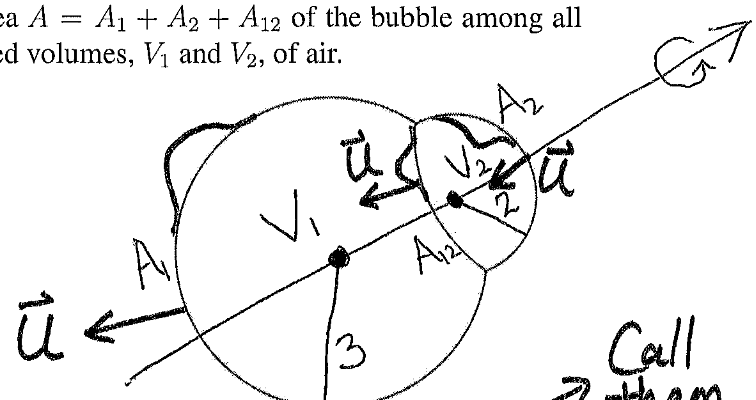
$$0 = A'(0) = -2\int_M H(H-\bar{H})\,dA$$

$$= -2\int_M (H)(H-\bar{H}) - \bar{H}(H-\bar{H})\cdot dA$$

$$= -2\int_M (H-\bar{H})^2\,dA \Rightarrow \underline{H = \bar{H} = \text{const.}}$$

(b) Now suppose that two round bubbles bump into each other, forming a "double bubble" as drawn. Double bubbles minimize the total area $A = A_1 + A_2 + A_{12}$ of the bubble among all configurations which enclose two separate fixed volumes, V_1 and V_2, of air.





Call them (H_1, H_{12}, H_2)

By part (a), each of the three sections of the double bubble must have constant mean curvature. In fact, each of the three sections of the double bubble are parts of spheres (unless $V_1 = V_2$ as in the picture on the left, in which case the middle section is a plane).

The image on the right is a cross section of a double bubble. Suppose the radius of the leftmost spherical portion is 3 and the radius of the rightmost spherical section is 2. What is the radius of the middle spherical portion?

Choose $\vec{u}$ pointing to the left on all 3 portions of the double bubble. Let $\eta = \vec{u} \cdot \vec{V}$ as before.

Label the 3 portions M_1 (left), M_2 (right), and M_{12} (center). Then to preserve volumes

$$\int_{M_1} \eta \, dA = \int_{M_{12}} \eta \, dA = \int_{M_2} \eta \, dA = I \text{ and}$$

$$0 = A'(0) = \int_{M_1} H_1 \eta \, dA + \int_{M_{12}} H_{12} \eta \, dA + \int_{M_2} H_2 \eta \, dA$$

$$= (H_1 + H_{12} + H_2) \cdot I, \text{ for all choices of } \eta \Rightarrow$$

$$0 = (H_1 + H_{12} + H_2)$$

$$= -\frac{1}{3} - \frac{1}{R} + \frac{1}{2} \longrightarrow \boxed{R = 6}$$