Midterm Exam, Math 421
Differential Geometry: Curves and Surfaces in $\mathbf{R}^3$

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Your Name: Solutions

Honor Pledge Signature:

Instructions: This is a 75 minute, closed book exam. You may bring one $8\frac{1}{2}'' \times 11''$ piece of paper with anything you've written on it to use during the exam, but nothing else. No collaboration on this exam is allowed. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points. Good luck on the exam!

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total	60	

Problem 1. Consider the surface M parametrized by

$$\vec{x}(u,v) = (1+u^3, 3-2v^5, u^3+v^5),$$

for $-\infty < u < \infty$ and $-\infty < v < \infty$.

(a) Compute $\vec{x}_u$, $\vec{x}_v$, and U.

$$\vec{X}_u = (3u^2, 0, 3u^2)$$

$$\vec{X}_v = (0, -10v^4, 5v^4)$$

$$\vec{X}_u \times \vec{X}_v = (30u^2v^4, -15u^2v^4, -30u^2v^4)$$

$$|\vec{X}_u \times \vec{X}_4| = 15u^2v^4|(2,-1,-2)| = 45u^2v^4$$

$$\vec{U} = \frac{X_u \times X_v}{|X_u \times X_v|} = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$$

(b) Compute $S_p(\vec{x}_u)$ and $S_p(\vec{x}_v)$.

$$S_p(v) = -\nabla_v U = 0 \quad \text{since } U = \text{constant vector}$$

Hence,

$$S_p(x_u) = -U_u = 0$$
$$S_p(x_v) = -U_v = 0.$$

(c) What is the shape operator $S_p(\vec{v})$, for any tangent vector $\vec{v}$?

$$S_p(v) = 0.$$

(d) What is the surface M? Describe it as well as you can.

Since $S_p(v) = 0$ everywhere, M is a plane with normal $\vec{U} = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$ that goes through the point $(1,3,0)$ when $u = v = 0$. Hence,

$$0 = \frac{2}{3}(x-1) - \frac{1}{3}(y-3) - \frac{2}{3}z \Rightarrow \boxed{2x - y - 2z = -1}$$

Problem 2.

(a) Give the definitions from class (and John Oprea's book "Differential Geometry and Its Applications") for the metric terms E, F, G and the shape operator terms l, m, n.

$$E = \vec{X}_u \cdot \vec{X}_u \qquad l = \vec{X}_u \cdot S_p(\vec{X}_u)$$
$$F = \vec{X}_u \cdot \vec{X}_v \qquad m = \vec{X}_u \cdot S_p(\vec{X}_v) = X_v \cdot S_p(X_u)$$
$$G = \vec{X}_v \cdot \vec{X}_v \qquad n = \vec{X}_v \cdot S_p(\vec{X}_v)$$

(b) Prove that x_u and x_v are eigenvectors of the shape operator at points where both $F = 0$ and $m = 0$.

$$S_p(\vec{X}_u) = a\vec{X}_u + b\vec{X}_v$$
$$S_p(\vec{X}_v) = c\vec{X}_u + d\vec{X}_v$$

since $\{\vec{X}_u, \vec{X}_v\}$ span the tangent plane.

$$0 = m = \vec{X}_v \cdot S_p(\vec{X}_u) = aF + bG = bG \Rightarrow b = 0$$
$$0 = m = \vec{X}_u \cdot S_p(X_v) = cE + dF = cE \Rightarrow c = 0.$$

Hence,

$$S_p(\vec{X}_u) = a\vec{X}_u$$
$$S_p(\vec{X}_v) = d\vec{X}_v,$$

so $\vec{X}_u$ and $\vec{X}_v$ are eigenvectors of S_p.

Problem 3.

(a) Define the Gauss curvature K of a surface in R^3.

$K = \det(S_p) = \det(A)$

$= ad - bc.$

(b) Define the mean curvature H of a surface in R^3.

$2H = \mathrm{tr}(S_p) = \mathrm{tr}(A)$

$= a + d.$

Given any basis vectors v, w, let

$S_p(v) = av + bw$

$S_p(w) = cv + dw$ $\Rightarrow$

$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

with respect to this basis.

We showed in class that these two definitions do not depend on the choice of basis $\{v, w\}$.

(c) Show that for any two linearly independent vectors v, w in the tangent plane T_pM to M at the point p,

$$S_p(v) \times S_p(w) = K(p) \;\; v \times w.$$

$S_p(v) \times S_p(w) = (av + bw) \times (cv + dw)$

$= ac\, v \times v + bc\, w \times v + ad\, v \times w + bd\, w \times w$

$= (ad - bc)\, v \times w$

$= K \cdot v \times w$

Remember:
$v \times v = 0$
$v \times w = -w \times v$

(d) Show that for any two linearly independent vectors v, w in the tangent plane T_pM to M at the point p,

$$S_p(v) \times w + v \times S_p(w) = 2H(p) \;\; v \times w.$$

$S_p(v) \times w + v \times S_p(w) = (av + bw) \times w + v \times (cv + dw)$

$= a\, v \times w + d\, v \times w$

$= (a + d)\, v \times w$

$= 2H \cdot v \times w$

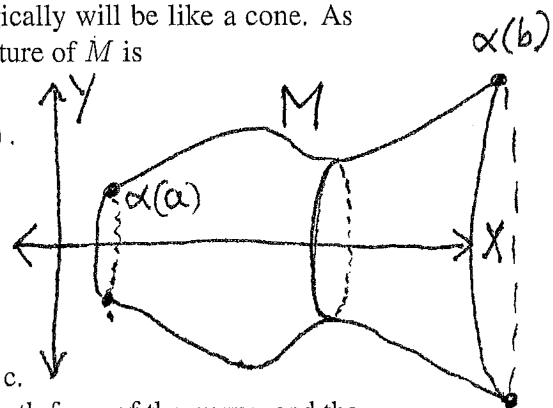
Problem 4. (The Gauss Bonnet Theorem for a Cone-like Surface of Revolution)

Let $\alpha(s) = (x(s), y(s))$, $a \le s \le b$, be a *unit speed* smooth curve with

$$\alpha'(a) = \alpha'(b) \longrightarrow \boxed{y'(a) = y'(b)} \qquad (1)$$

in the xy plane which does not intersect itself or the x-axis. Let M be the surface of revolution created by rotating the curve α around the x-axis, which topologically will be like a cone. As derived on page 119 of the book, the formula for the Gauss curvature of M is

$$K = \frac{x'(s)}{y(s)}\left(x''(s)y'(s) - y''(s)x'(s)\right).$$



Using the above formula, prove that

$$\int_M K\,dA = c$$

for any such curve α, for some constant c. Compute the constant c.

Hints: You will need to use $dA = 2\pi y\,ds$, where ds is the length form of the curve, and the fact that $\boxed{x'(s)^2 + y'(s)^2 = 1.}$ $\longrightarrow 2x'(s)x''(s) + 2y'(s)y''(s) = 0 \longrightarrow$

$$\boxed{x'x'' = -y'y''}$$

$$\int_M K\,dA = \int_a^b \frac{x'}{y}\left(x''y' - y''x'\right)\cdot 2\pi y\,ds$$

$$= \int_a^b 2\pi\left(\underline{\underline{x'x''}}\,y' - y''(x')^2\right)ds$$

$$= \int_a^b 2\pi\left(-y'y''y' - y''(x')^2\right)ds$$

$$= \int_a^b -2\pi y''(s)\left((y')^2 + (x')^2\right)ds$$

$$= \int_a^b -2\pi y''(s)\,ds$$

$$= -2\pi y'(s)\Big]_a^b = -2\pi\left(y'(b) - y'(a)\right) = \boxed{0.}$$

Problem 5. Let U be the unit normal to the **smooth** surface M at the point $p \in M$, and let u_1 and u_2 be an orthonormal basis for the tangent plane T_pM to M at p. Define

$$u(\theta) = \cos(\theta)u_1 + \sin(\theta)u_2,$$

and let P_θ be the plane through p containing the vectors U and $u(\theta)$. Suppose that

- $M \cap P_0$ is a straight line,
- $M \cap P_{\pi/4}$ is a straight line, and
- $M \cap P_{\pi/2}$ is a straight line.

The normal curvatures $k(u(0)) = k(u(\pi/4)) = k(u(\pi/2)) = 0$.

Prove that the shape operator of M at the point p equals zero, that is,

$$S_p(v) = 0,$$

for *all* tangent vectors v.

$$0 = k(u(0)) = k(u_1) = u_1 \cdot S(u_1) = l$$

$$0 = k(u(\tfrac{\pi}{2})) = k(u_2) = u_2 \cdot S(u_2) = n$$

$$0 = k(u(\pi/4)) = k\left(\tfrac{1}{\sqrt{2}}u_1 + \tfrac{1}{\sqrt{2}}u_2\right)$$

$$= \left(\tfrac{1}{\sqrt{2}}u_1 + \tfrac{1}{\sqrt{2}}u_2\right) \cdot S\left(\tfrac{1}{\sqrt{2}}u_1 + \tfrac{1}{\sqrt{2}}u_2\right)$$

$$= \tfrac{1}{2}(u_1 \cdot S(u_1)) + \tfrac{1}{2}(u_2 \cdot S(u_2)) + \tfrac{1}{2}(u_1 \cdot S(u_2) + u_2 \cdot S(u_1))$$

$$\Rightarrow 0 = u_1 \cdot (S(u_2)) + u_2 \cdot S(u_1)$$

$$\Rightarrow 0 = u_1 \cdot S(u_2) = u_2 \cdot S(u_1) = m$$

Let $S(u_1) = au_1 + bu_2$

$$0 = u_1 \cdot S(u_1) = a \cdot 1 + b \cdot 0 = a$$

$$0 = u_2 \cdot S(u_1) = a \cdot 0 + b \cdot 1 = b$$

$\Rightarrow$ $\boxed{S(u_1) = \vec{0}. \text{ Similarly, } S(u_2) = \vec{0}.}$

Hence, $S_p(v) = \vec{0}$ for all v.