

# A New Monotonic and Clone-Independent Single-Winner Election Method

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## 1 Introduction

In 1997, I proposed to a large number of people who are interested in mathematical aspects of election methods a new method that satisfies Pareto, monotonicity, resolvability, independence of clones, reversal symmetry, Smith-IIA, and Schwartz. This method immediately attracted a lot of attention and very many enthusiastic supporters. Today, this method is promoted e.g. by Diana Galletly [1], Mathew Goldstein [2], Jobst Heitzig [3], Raul Miller, Mike Ossipoff [4], Russ Paielli, Norman Petry, Manoj Srivastava, and Anthony Towns and it is analyzed e.g. in the websites of Blake Cretney [5], Steve Eppley [6], Eric Gorr [7], and Rob LeGrand [8]. Today, this method is taught e.g. by James E. Falk of George Washington University and Thomas K. Yan of Cornell University [9]. In January 2003, the board of Software in the Public Interest (SPI) adopted this method unanimously [10]. In June 2003, the DEBIAN Project adopted this method with 144 against 16 votes [11, 12]. Therefore, a more detailed motivation and explanation of the method is overdue.

There has been some debate about an appropriate name for the method. Some people suggested names like “Beatpath Method”, “Beatpath Winner”, “Path Voting”, “Schwartz Sequential Dropping” (SSD) or “Cloneproof Schwartz Sequential Dropping” (CSSD or CpSSD). However, I prefer the name “Schulze method”, not because of academic arrogance, but because the other names do not refer to the method itself but to specific heuristics for implementing it, and so

may mislead readers into believing that no other method for implementing it is possible. In my opinion, although it is advantageous to have an intuitive and convincing heuristic, in the end only the properties of the method are relevant.

I have already found some implementations of my method in the internet. Unfortunately, most implementations that I have seen were inefficient because the programmers have not understood the Floyd algorithm so that the implementations had a runtime of  $O(N^5)$  although the winners of this method can be calculated in a runtime of  $O(N^3)$ , where  $N$  is the number of candidates.

It is presumed that each voter casts at least a *partial ranking* of all candidates. That means: It is presumed that for each voter  $V$  the relation “voter  $V$  strictly prefers candidate  $A$  to candidate  $B$ ” is irreflexive, asymmetric, and transitive on the set of candidates. But it is not presumed that each voter casts a *complete ranking*. That means: It is not presumed that this relation is also linear.

Suppose that  $d[X,Y]$  is the number of voters who strictly prefer candidate  $X$  to candidate  $Y$ . Then the *Smith set* is the smallest non-empty set of candidates with  $d[A,B] > d[B,A]$  for each candidate  $A$  of this set and each candidate  $B$  outside this set. *Smith-IIA* (where IIA means Independence from Irrelevant Alternatives) says that adding a candidate who is not in the new Smith set should not change the probability that a given and already running candidate is elected. Smith-IIA implies the majority criterion for solid coalitions and the Condorcet criterion. Unfortunately, compliance with the Condorcet criterion implies violation of other desired criteria like participation [13], later-no-harm, and later-no-help [14].

A *chain from candidate  $A$  to candidate  $B$*  is an ordered set of candidates  $C(1), \dots, C(n)$  with the following

three properties:

1.  $C(1)$  is identical to  $A$ .
2.  $C(n)$  is identical to  $B$ .
3.  $d[C(i),C(i+1)] - d[C(i+1),C(i)] > 0$  for each  $i = 1, \dots, (n-1)$ .

A *Schwartz winner* is a candidate  $A$  who has chains at least to every other candidate  $B$  who has a chain to candidate  $A$ . The *Schwartz set* is the set of all Schwartz winners. *Schwartz* says that the winner must be a Schwartz winner.

In section 2, the Schulze method is defined. In section 3, well-definedness of this method is proven. In section 4, I present an implementation with a runtime of  $O(N^3)$ . In section 5, I prove that this method satisfies Pareto, monotonicity, resolvability, independence of clones, and reversal symmetry. From the definition of the Schulze method, it is clear that this method meets Smith-IIA and Schwartz.

Another election method that satisfies Pareto, monotonicity, resolvability, independence of clones, reversal symmetry, Schwartz, and Smith-IIA is Tideman's Ranked Pairs method [15, 16]. However, appendix A demonstrates that the proposed method is not identical with the Ranked Pairs method. Appendix B demonstrates that the proposed method can violate the participation criterion in a very drastic manner. A special provision of the implementation used by SPI and DEBIAN is described in appendix C. Appendix D explains how the proposed method can be interpreted as a method where successively the weakest pairwise defeats are "eliminated." Appendix E presents a concrete example where the proposed method does not find a unique winner.

## 2 Definition of the Schulze Method

**Stage 1:** Suppose that  $d[A,B]$  is the number of voters who strictly prefer candidate  $A$  to candidate  $B$ .

A *path from candidate  $A$  to candidate  $B$*  is an ordered set of candidates  $C(1), \dots, C(n)$  with the following two properties:

1.  $C(1)$  is identical to  $A$ .
2.  $C(n)$  is identical to  $B$ .

The *strength* of the path  $C(1), \dots, C(n)$  is  $\min \{ d[C(i),C(i+1)] - d[C(i+1),C(i)] \mid i = 1, \dots, (n-1) \}$ .

Thus a chain from candidate  $A$  to candidate  $B$ , as defined in the introduction, is simply a path with positive strength.

$p[A,B] := \max \{ \min \{ d[C(i),C(i+1)] - d[C(i+1),C(i)] \mid i = 1, \dots, (n-1) \} \mid C(1), \dots, C(n) \text{ is a path from candidate } A \text{ to candidate } B \}$ .

In other words:  $p[A,B]$  is the strength of the strongest path from candidate  $A$  to candidate  $B$ .

Candidate  $A$  is a *potential winner* if and only if  $p[A,B] \geq p[B,A]$  for every other candidate  $B$ .

When  $p[A,B] > p[B,A]$ , then we say: "Candidate  $A$  disqualifies candidate  $B$ ".

**Stage 2:** If there is only one potential winner, then this potential winner is the *unique winner*. If there is more than one potential winner, then a *Tie-Breaking Ranking of the Candidates* (TBRC) is calculated as follows:

1. Pick a random ballot and use its rankings; consider ties as unsorted with regard to each other.
2. Continue picking ballots randomly from those that have not yet been picked. When you find one that orders previously unsorted candidates, use the ballot to sort them. Do not change the order of the already sorted.
3. If you go through all ballots, and some candidates are still not sorted, order them randomly.

The winner is that potential winner who is ranked highest in this TBRC.

## 3 Well-Definedness

On first view, it is not clear whether the Schulze method is well defined. It seems to be possible that candidates disqualify each other in such a manner that there is no candidate  $A$  with  $p[A,B] \geq p[B,A]$  for every other candidate  $B$ . However, the following proof demonstrates that path defeats are transitive. That means: When candidate  $A$  disqualifies candidate  $B$  and when candidate  $B$  disqualifies candidate  $C$ , then also candidate  $A$  disqualifies candidate  $C$ .

Claim:  $( p[A,B] > p[B,A] \text{ and } p[B,C] > p[C,B] ) \Rightarrow p[A,C] > p[C,A]$ .

**Proof:** Suppose

- (1)  $p[A,B] > p[B,A]$  and
- (2)  $p[B,C] > p[C,B]$ .

The following statements are valid:

- (3)  $\min \{ p[A,B]; p[B,C] \} \leq p[A,C]$ .
- (4)  $\min \{ p[A,C]; p[C,B] \} \leq p[A,B]$ .
- (5)  $\min \{ p[B,A]; p[A,C] \} \leq p[B,C]$ .
- (6)  $\min \{ p[B,C]; p[C,A] \} \leq p[B,A]$ .
- (7)  $\min \{ p[C,A]; p[A,B] \} \leq p[C,B]$ .
- (8)  $\min \{ p[C,B]; p[B,A] \} \leq p[C,A]$ .

For example: If  $\min \{ p[A,B]; p[B,C] \}$  was strictly larger than  $p[A,C]$ , then this would be a contradiction to the definition of  $p[A,C]$  since there would be a route from candidate A to candidate C via candidate B with a strength of more than  $p[A,C]$ ; and if this route was not itself a path (because it passed through some candidates more than once) then some subset of its links would form a path from candidate A to candidate C with a strength of more than  $p[A,C]$ .

**Case 1:** Suppose

- (9a)  $p[A,B] \geq p[B,C]$ .
- Combining (2) and (9a) gives:
- (10a)  $p[A,B] > p[C,B]$ .
- Combining (7) and (10a) gives:
- (11a)  $p[C,A] \leq p[C,B]$ .
- Combining (3) and (9a) gives:
- (12a)  $p[B,C] \leq p[A,C]$ .
- Combining (11a), (2), and (12a) gives:
- (13a)  $p[C,A] \leq p[C,B] < p[B,C] \leq p[A,C]$ .

**Case 2:** Suppose

- (9b)  $p[A,B] < p[B,C]$ .
- Combining (1) and (9b) gives:
- (10b)  $p[B,C] > p[B,A]$ .
- Combining (6) and (10b) gives:
- (11b)  $p[C,A] \leq p[B,A]$ .
- Combining (3) and (9b) gives:
- (12b)  $p[A,B] \leq p[A,C]$ .
- Combining (11b), (1), and (12b) gives:
- (13b)  $p[C,A] \leq p[B,A] < p[A,B] \leq p[A,C]$ .

Therefore, the relation defined by  $p[A,B] > p[B,A]$  is transitive.

## 4 Implementation

The strength of the strongest path  $p[i,j]$  from candidate  $i$  to candidate  $j$  can be calculated with the Floyd algorithm [17]. The runtime to calculate the strengths of all paths is  $O(N^3)$ . It cannot be said frequently enough

that the order of the indices in the triple-loop of the Floyd algorithm is **not** irrelevant.

Input:  $d[i,j]$  with  $i \neq j$  is the number of voters who strictly prefer candidate  $i$  to candidate  $j$ .

Output: “ $w[i] = \text{true}$ ” means that candidate  $i$  is a potential winner. “ $w[i] = \text{false}$ ” means that candidate  $i$  is not a potential winner.

```
for i := 1 to N do
for j := 1 to N do
  if ( i ≠ j ) then
    p[i,j] := d[i,j] - d[j,i];
```

```
for i := 1 to N do
for j := 1 to N do
  if ( i ≠ j ) then
    for k := 1 to N do
      if ( i ≠ k ) then
        if ( j ≠ k ) then
          {
            s := min { p[j,i], p[i,k] };
            if ( p[j,k] < s ) then
              p[j,k] := s ;
          }
```

```
for i := 1 to N do
  {
    w[i] := true ;
    for j := 1 to N do
      if ( i ≠ j ) then
        if ( p[j,i] > p[i,j] ) then
          w[i] := false ;
  }
```

## 5 Properties

### 5.1 Pareto

*Pareto* says that when no voter strictly prefers candidate B to candidate A and at least one voter strictly prefers candidate A to candidate B then candidate B must not be elected.

The Schulze method meets Pareto.

**Proof:** Suppose no voter strictly prefers candidate B to candidate A and at least one voter strictly prefers candidate A to candidate B. Then  $d[A,B] > 0$  and  $d[B,A] = 0$ .

**Case 1:** If BA is already the strongest path from candidate B to candidate A, then  $p[B,A] = d[B,A] - d[A,B] < 0$ . Therefore, candidate A disqualifies candidate B because  $p[A,B] \geq d[A,B] - d[B,A] > 0$ , so that  $p[A,B] > p[B,A]$ .

**Case 2:** Suppose that  $B,C(1),\dots,C(n),A$  is the strongest path from candidate B to candidate A. As every voter who strictly prefers candidate B to candidate C(1) also necessarily strictly prefers candidate A to candidate C(1), we get  $d[A,C(1)] \geq d[B,C(1)]$ . As every voter who strictly prefers candidate C(1) to candidate A also necessarily strictly prefers candidate C(1) to candidate B, we get  $d[C(1),B] \geq d[C(1),A]$ . Therefore,  $d[A,C(1)] - d[C(1),A] \geq d[B,C(1)] - d[C(1),B]$ . For the same reason, we get  $d[C(n),B] - d[B,C(n)] \geq d[C(n),A] - d[A,C(n)]$ . Therefore, the path  $A,C(1),\dots,C(n),B$  is at least as strong as the path  $B,C(1),\dots,C(n),A$ . In so far as  $B,C(1),\dots,C(n),A$  is the strongest path from candidate B to candidate A by presumption, we get  $p[A,B] \geq p[B,A]$ .

Suppose that candidate B is a potential winner. Then also candidate A is a potential winner.

**Proof:** Suppose that  $B,C(1),\dots,C(n),X$  is the strongest path from candidate B to candidate X. Then,  $A,C(1),\dots,C(n),X$  is a path, but not necessarily the strongest path, from candidate A to candidate X with at least the same strength because  $d[A,C(1)] - d[C(1),A] \geq d[B,C(1)] - d[C(1),B]$ . Therefore,  $p[A,X] \geq p[B,X]$  for every candidate X other than candidate A or candidate B. Suppose that  $X,C(1),\dots,C(n),A$  is the strongest path from candidate X to candidate A. Then,  $X,C(1),\dots,C(n),B$  is a path, but not necessarily the strongest path, from candidate X to candidate B with at least the same strength because  $d[C(n),B] - d[B,C(n)] \geq d[C(n),A] - d[A,C(n)]$ . Therefore,  $p[X,B] \geq p[X,A]$  for every candidate X other than candidate A or candidate B.

Since candidate B is a potential winner,  $p[B,X] \geq p[X,B]$  for every other candidate X. With  $p[A,X] \geq p[B,X]$ ,  $p[B,X] \geq p[X,B]$ , and  $p[X,B] \geq p[X,A]$ , we get  $p[A,X] \geq p[X,A]$  for every other candidate X. Therefore, also candidate A is a potential winner.

Therefore, when no voter strictly prefers candidate B to candidate A and at least one voter strictly prefers candidate A to candidate B then when candidate B is a potential winner also candidate A is

a potential winner. Therefore, candidate B cannot be elected at stage 1 of the Schulze method. Candidate B cannot be elected at stage 2, either, since candidate A is necessarily ranked above candidate B in the TBRC.

## 5.2 Monotonicity

*Monotonicity* says that when some voters rank candidate A higher without changing the order in which they rank the other candidates relatively to each other then the probability that candidate A is elected must not decrease.

The Schulze method meets monotonicity.

**Proof:** Suppose candidate A was a potential winner. Then  $p_{old}[A,B] \geq p_{old}[B,A]$  for every other candidate B.

**Part 1:** Suppose some voters rank candidate A higher without changing the order in which they rank the other candidates. Then  $d_{new}[A,X] \geq d_{old}[A,X]$  and  $d_{new}[X,A] \leq d_{old}[X,A]$  for every other candidate X.  $d_{new}[X,Y] = d_{old}[X,Y]$  when neither candidate X nor candidate Y is identical to candidate A. Therefore  $d_{new}[A,X] - d_{new}[X,A] \geq d_{old}[A,X] - d_{old}[X,A]$  for every other candidate X. And  $d_{new}[X,Y] - d_{new}[Y,X] = d_{old}[X,Y] - d_{old}[Y,X]$  when neither candidate X nor candidate Y is identical to candidate A. For every candidate B other than candidate A the value  $p[A,B]$  can only increase but not decrease with  $d[A,X] - d[X,A]$  since only AX but not XA can be in the strongest path from candidate A to candidate B and the value  $p[B,A]$  can only decrease but not increase with  $d[A,X] - d[X,A]$  since only XA but not AX can be in the strongest path from candidate B to candidate A. Therefore  $p_{new}[A,B] \geq p_{old}[A,B]$  and  $p_{new}[B,A] \leq p_{old}[B,A]$ . Therefore  $p_{new}[A,B] \geq p_{new}[B,A]$  so that candidate A is still a potential winner.

**Part 2:** Suppose that candidate E is not identical to candidate A. It remains to be proven that when candidate E was not a potential winner before then he is still not a potential winner. Suppose that candidate E was not a potential winner. Then there must have been a candidate F other than candidate E with

$$(1) p_{old}[F,E] > p_{old}[E,F].$$

Then, of course, also  $p_{new}[F,E] > p_{new}[E,F]$  is valid unless XA was a weakest link in the

strongest path from candidate F to candidate E and/or AY was the weakest link in the strongest path from candidate E to candidate F. Without loss of generality, we can presume that candidate F is not identical to candidate A and that

$$(2) p_{old}[A,E] = p_{old}[E,A]$$

because otherwise with  $p_{old}[A,E] > p_{old}[E,A]$  we would immediately get  $p_{new}[A,E] > p_{new}[E,A]$  (because of the considerations in Part 1) so that we would immediately get that candidate E is still not a potential winner. Since candidate A was a potential winner, we get

$$(3) p_{old}[A,F] \geq p_{old}[F,A].$$

The following statements are valid for the same reason as in section 3:

$$(4) \min \{ p_{old}[A,E]; p_{old}[E,F] \} \leq p_{old}[A,F].$$

$$(5) \min \{ p_{old}[A,F]; p_{old}[F,E] \} \leq p_{old}[A,E].$$

$$(6) \min \{ p_{old}[E,A]; p_{old}[A,F] \} \leq p_{old}[E,F].$$

$$(7) \min \{ p_{old}[E,F]; p_{old}[F,A] \} \leq p_{old}[E,A].$$

$$(8) \min \{ p_{old}[F,A]; p_{old}[A,E] \} \leq p_{old}[F,E].$$

$$(9) \min \{ p_{old}[F,E]; p_{old}[E,A] \} \leq p_{old}[F,A].$$

**Case 1:** Suppose XA was a weakest link in the strongest path from candidate F to candidate E. Then

$$(10a) p_{old}[F,E] = p_{old}[F,A] \text{ and}$$

$$(11a) p_{old}[A,E] \geq p_{old}[F,E].$$

Now (3), (10a), and (1) give

$$(12a) p_{old}[A,F] \geq p_{old}[F,A] = p_{old}[F,E] > p_{old}[E,F],$$

while (2), (11a), and (1) give

$$(13a) p_{old}[E,A] = p_{old}[A,E] \geq p_{old}[F,E] > p_{old}[E,F].$$

But (12a) and (13a) together contradict (6).

**Case 2:** Suppose AY was the weakest link in the strongest path from candidate E to candidate F. Then

$$(10b) p_{old}[E,F] = p_{old}[A,F] \text{ and}$$

$$(11b) p_{old}[E,A] > p_{old}[E,F].$$

Now (11b), (10b), and (3) give

$$(12b) p_{old}[E,A] > p_{old}[E,F] = p_{old}[A,F] \geq p_{old}[F,A],$$

while (1), (10b), and (3) give

$$(13b) p_{old}[F,E] > p_{old}[E,F] = p_{old}[A,F] \geq p_{old}[F,A].$$

But (12b) and (13b) together contradict (9).

**Conclusion:** When some voters rank candidate A higher without changing the order in which they rank

the other candidates relatively to each other, then (a) when candidate A was a potential winner candidate A is still a potential winner and (b) every other candidate E who was not a potential winner is still not a potential winner and (c) candidate A can only increase in the TBRC while the positions of the other candidates are not changed relatively to each other. Therefore, the probability that candidate A is elected cannot decrease.

### 5.3 Resolvability

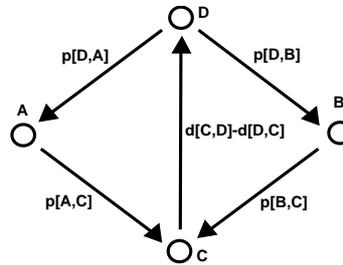
*Resolvability* says that at least in those cases in which there are no pairwise ties and there are no pairwise defeats of equal strength the winner must be unique.

The Schulze method meets resolvability.

**Proof:** Suppose that there is no unique winner. Suppose that candidate A and candidate B are potential winners. Then:

$$(1) p[A,B] = p[B,A].$$

Suppose that there are no pairwise ties and that there are no pairwise defeats of equal strength. Then  $p[A,B] = p[B,A]$  means that the weakest link in the strongest path from candidate A to candidate B and the weakest link in the strongest path from candidate B to candidate A must be the same link, say CD. Then this situation looks as follows:



As the weakest link of the strongest path from candidate B to candidate A is CD, we get:

$$(2) p[D,A] > p[B,A].$$

As the weakest link of the strongest path from candidate A to candidate B is CD, we get:

$$(3) p[A,D] = p[A,B].$$

With (2), (1), and (3) we get:

$$(4) p[D,A] > p[B,A] = p[A,B] = p[A,D] \text{ which contradicts the presumption that candidate A is a potential winner.}$$

### 5.4 Independence of Clones

An election method is *independent of clones* if the following holds:

Suppose that candidate D and candidate E are two different candidates.

1. Suppose (a) that there is at least one voter who either strictly prefers candidate D to candidate E or strictly prefers candidate E to candidate D or (b) that candidate D is elected with zero probability.
2. Suppose that candidate D is replaced by a set of candidates  $D(1), \dots, D(m)$  in such a manner that for every candidate  $D(i)$  in this set, for every candidate F outside this set, and for every voter V the following two statements are valid:
  - a) V strictly preferred D to F  $\Leftrightarrow$  V strictly prefers  $D(i)$  to F.
  - b) V strictly preferred F to D  $\Leftrightarrow$  V strictly prefers F to  $D(i)$ .

Then the probability that candidate E is elected must not change.

The Schulze method is independent of clones.

**Proof:** Suppose that candidate D is replaced by a set of candidates  $D(1), \dots, D(m)$  in the manner described above. Then  $d_{new}[A, D(i)] = d_{old}[A, D]$  for every candidate A outside the set  $D(1), \dots, D(m)$  and for every  $i = 1, \dots, m$ . And  $d_{new}[D(i), B] = d_{old}[D, B]$  for every candidate B outside the set  $D(1), \dots, D(m)$  and for every  $i = 1, \dots, m$ .

(1) Case 1: Suppose that the strongest path  $C(1), \dots, C(n)$  from candidate A to candidate B did not contain candidate D. Then  $C(1), \dots, C(n)$  is still a path from candidate A to candidate B with the same strength. Therefore:  $p_{new}[A, B] \geq p_{old}[A, B]$ .

Case 2: Suppose that the strongest path  $C(1), \dots, C(n)$  from candidate A to candidate B contained candidate D. Then  $C(1), \dots, C(n)$  with D replaced by an arbitrarily chosen candidate  $D(i)$  is still a path from candidate A to candidate B with the same strength. Therefore:  $p_{new}[A, B] \geq p_{old}[A, B]$ .

(2) Case 1: Suppose that the strongest path  $C(1), \dots, C(n)$  from candidate A to candidate B does not contain candidates of the set  $D(1), \dots, D(m)$ . Then  $C(1), \dots, C(n)$  was a path from candidate A to candidate B with the same strength. Therefore:  $p_{old}[A, B] \geq p_{new}[A, B]$ .

Case 2: Suppose that the strongest path  $C(1), \dots, C(n)$  from candidate A to candidate B contains some candidates of the set  $D(1), \dots, D(m)$ . Then  $C(1), \dots, C(n)$  where the part of this path from the first occurrence of a candidate of the set  $D(1), \dots, D(m)$  to the last occurrence of a candidate of the set  $D(1), \dots, D(m)$  is replaced by

candidate D was a path from candidate A to candidate B with at least the same strength. Therefore:  $p_{old}[A, B] \geq p_{new}[A, B]$ .

With (1) and (2), we get:  $p_{new}[A, B] = p_{old}[A, B]$ .

When we set  $A \equiv D$  in (1) and (2), we get:  $p_{new}[D(i), B] = p_{old}[D, B]$  for every candidate B outside the set  $D(1), \dots, D(m)$  and for every  $i = 1, \dots, m$ .

When we set  $B \equiv D$  in (1) and (2), we get:  $p_{new}[A, D(i)] = p_{old}[A, D]$  for every candidate A outside the set  $D(1), \dots, D(m)$  and for every  $i = 1, \dots, m$ .

Suppose candidate A, who is not identical to candidate D, was a potential winner, then  $p_{old}[A, B] \geq p_{old}[B, A]$  for every other candidate B; because of the above considerations we get  $p_{new}[A, B] \geq p_{new}[B, A]$  for every other candidate B; therefore, candidate A is still a potential winner. Suppose candidate B, who is not identical to candidate D, was not a potential winner, then  $p_{old}[B, A] < p_{old}[A, B]$  for at least one other candidate A; because of the above considerations we get  $p_{new}[B, A] < p_{new}[A, B]$  for at least this other candidate A; therefore, candidate B is still not a potential winner.

Presumption 1 in the definition of independence of clones guarantees that at least in those situations in which the TBRC has to be used to choose from the candidates  $D(1), \dots, D(m), E$  (a) candidate E is ranked above each of the candidates  $D(1), \dots, D(m)$  when he was originally ranked above candidate D. (b) candidate E is ranked below each of the candidates  $D(1), \dots, D(m)$  when he was originally ranked below candidate D. Therefore, replacing candidate D by a set of candidates  $D(1), \dots, D(m)$  can neither change whether candidate E is a potential winner nor, when the TBRC has to be used, where this candidate is ranked in the TBRC.

## 5.5 Reversal Symmetry

*Reversal symmetry* says that when candidate A is the unique winner then when the individual preferences of each voter are inverted then candidate A must not be elected.

The Schulze method meets reversal symmetry.

**Proof:** Suppose candidate A was the unique winner. Then there must have been at least one other candidate B with  $p_{old}[A, B] > p_{old}[B, A]$ . (Since the relation defined by  $p[X, Y] > p[Y, X]$  is transitive there must have been at least one candidate B other than candidate A with  $p[B, E] \geq p[E, B]$  for every candidate E other than candidate A or candidate B. Since candidate A was the unique winner and since no candidate other than

candidate A has disqualified candidate B, candidate A must have disqualified candidate B, i.e.  $p_{old}[A,B] > p_{old}[B,A]$ .)

When the individual preferences of each voter are inverted then  $d_{new}[Y,X] = d_{old}[X,Y]$  for each pair XY of candidates. When  $C(1), \dots, C(n)$  was a path from candidate X to candidate Y of strength Z then  $C(n), \dots, C(1)$  is a path from candidate Y to candidate X of strength Z. Therefore,  $p_{new}[Y,X] = p_{old}[X,Y]$  for each pair XY of candidates. Therefore,  $p_{new}[B,A] > p_{new}[A,B]$  so that candidate B disqualifies candidate A.

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## A Tideman's Ranked Pairs Method

Tideman's Ranked Pairs method [15, 16] is very similar to my method in so far as both methods meet Pareto, monotonicity, resolvability, independence of clones, reversal symmetry, Schwartz and Smith-IIA. However, the following example demonstrates that these methods are not identical.

Example:

3	ACDB
5	ADBC
4	BACD
5	BCDA
2	CADB
5	CDAB
2	DABC
4	DBAC

The matrix  $d[i,j]$  of pairwise defeats looks as follows:

	A	B	C	D
A	—	17	18	14
B	13	—	20	9
C	12	10	—	19
D	16	21	11	—

The matrix  $p[i,j]$  of the path strengths looks as follows:

	A	B	C	D
A	—	6	6	6
B	2	—	10	8
C	2	8	—	8
D	2	12	10	—

Candidate A is the unique Schulze winner because candidate A is the unique candidate with  $p[A,X] \geq p[X,A]$  for every other candidate X.

Tideman suggests to take successively the strongest pairwise defeat and to lock it if it does not create a directed cycle with already locked pairwise defeats or to skip it if it would create a directed cycle with already locked pairwise defeats. The winner of the Ranked Pairs method is that candidate X who wins each pairwise comparison which is locked and in which candidate X is involved.

Tideman's Ranked Pairs method locks  $D > B$ . Then it locks  $B > C$ . Then it skips  $C > D$  since it would create a directed cycle with the already locked defeats  $D > B$  and  $B > C$ . Then it locks  $A > C$ . Then it locks  $A > B$ . Then it locks  $D > A$ . Thus, the Ranked Pairs winner is candidate D.

## B The Participation Criterion

The *participation* criterion says that adding a set of identical ballots on which candidate A is strictly preferred to candidate B should not change the winner from candidate A to candidate B. Moulin [13] proved that the Condorcet criterion and the participation criterion are incompatible. Pérez [18] demonstrated that most Condorcet methods can violate the participation criterion in a very drastic manner. That means: It can happen that adding a set of identical ballots on which candidate A is strictly preferred to every other candidate changes the winner from candidate A to another candidate or that adding a set of identical ballots on which every other candidate is strictly preferred to candidate B changes the winner from another candidate to candidate B. The following example demonstrates that also the Schulze method can violate the participation criterion in a very drastic manner. (The basic idea for this example came from Blake Cretney.)

Example:

4	ABCDEF
2	ABFDEC
4	AEBFCD
2	AEFBCD
2	BFACDE
2	CDBEFA
4	CDBFEA
12	DECABF
8	ECDBFA
10	FABCDE

6	FABDEC
4	FEDBCA

The matrix  $d[i,j]$  of pairwise defeats looks as follows:

	A	B	C	D	E	F
A	—	40	30	30	30	24
B	20	—	34	30	30	38
C	30	26	—	36	22	30
D	30	30	24	—	42	30
E	30	30	38	18	—	32
F	36	22	30	30	28	—

The matrix  $p[i,j]$  of the path strengths looks as follows:

	A	B	C	D	E	F
A	—	20	8	8	8	16
B	12	—	8	8	8	16
C	4	4	—	12	12	4
D	4	4	16	—	24	4
E	4	4	16	12	—	4
F	12	12	8	8	8	—

Candidate A is the unique winner since he is the only candidate with  $p[A,X] \geq p[X,A]$  for every other candidate X. However, when 3 AEFCBD ballots are added then the matrix  $d[i,j]$  of pairwise defeats looks as follows:

	A	B	C	D	E	F
A	—	43	33	33	33	27
B	20	—	34	33	30	38
C	30	29	—	39	22	30
D	30	30	24	—	42	30
E	30	33	41	21	—	35
F	36	25	33	33	28	—

The matrix  $p[i,j]$  of the path strengths looks as follows:

	A	B	C	D	E	F
A	—	23	5	5	5	13
B	9	—	5	5	5	13
C	7	7	—	15	15	7
D	7	7	19	—	21	7
E	7	7	19	15	—	7
F	9	9	5	5	5	—

Now, candidate D is the unique winner since he is the only candidate with  $p[D,X] \geq p[X,D]$  for every other candidate X. Thus the 3 AEFCBD voters change the winner from candidate A to candidate D.

### C A Special Provision of the Implementation used by SPI and DEBIAN

There has been some debate about how to measure the strength of a pairwise defeat when it is presumed that on the one side each voter has a sincere complete ranking of all candidates, but on the other side some voters vote only a partial ranking because of strategical considerations. I suggest that then the strength of a pairwise defeat should be measured primarily by the absolute number of votes for the winner of this pairwise defeat and secondarily by the margin of this pairwise defeat. The purpose of this provision is to give an additional incentive to the voters to give different preferences to candidates to which the voters would have given the same preference because of strategical considerations otherwise.

The resulting version of this method is used by SPI and DEBIAN because (a) here the number of candidates is usually very small and the voters are usually well-informed about the different candidates so that it can be presumed that each voter has a sincere complete ranking of all candidates and (b) here the number of voters is usually very small and the voters are usually well-informed about the opinions of the other voters so that the incentive to cast only a partial ranking because of strategical considerations is large.

The resulting version still satisfies Pareto, monotonicity, resolvability, independence of clones, reversal symmetry, Smith-IIA, and Schwartz. When each voter casts a complete ranking then this version is identical to the version defined in section 2. I suggest that in the general case the version as defined in section 2 should be used. Only in situations similar to the above described situation in SPI and DEBIAN, the version as defined in this appendix should be used.

When the strength of a pairwise defeat is measured primarily by  $p1$  (= the absolute number of votes for the winner of this pairwise defeat) and secondarily by  $p2$  (= the margin of this pairwise defeat), then a possible implementation looks as follows:

Input:  $d[i,j]$  with  $i \neq j$  is the number of voters who strictly prefer candidate  $i$  to candidate  $j$ .

Output: “ $w[i] = \text{true}$ ” means that candidate  $i$  is a potential winner. “ $w[i] = \text{false}$ ” means that candidate  $i$  is not a potential winner.

```
for i := 1 to N do
for j := 1 to N do
```

```
    if ( i ≠ j ) then
    {
    p2[i,j] := d[i,j] - d[j,i] ;
    if ( d[i,j] > d[j,i] ) then
    p1[i,j] := d[i,j] ;
    if ( d[i,j] ≤ d[j,i] ) then
    p1[i,j] := -1 ;
    }

for i := 1 to N do
for j := 1 to N do
    if ( i ≠ j ) then
    for k := 1 to N do
    if ( i ≠ k ) then
    if ( j ≠ k ) then
    {
    s := min { p1[j,i], p1[i,k] } ;
    t := min { p2[j,i], p2[i,k] } ;
    if ( ( p1[j,k] < s ) or ( ( p1[j,k] = s ) and
    ( p2[j,k] < t ) ) ) then
    {
    p1[j,k] := s ;
    p2[j,k] := t ;
    }
    }

for i := 1 to N do
{
w[i] := true ;
for j := 1 to N do
    if ( i ≠ j ) then
    if ( ( p1[j,i] > p1[i,j] ) or ( ( p1[j,i] = p1[i,j] ) and
    ( p2[j,i] > p2[i,j] ) ) ) then
    w[i] := false ;
}
```

### D The Schwartz Set Heuristic

Another way of looking at the proposed method is to interpret it as a method where successively the weakest pairwise defeats are “eliminated”. The formulation of this method then becomes very similar to Condorcet’s original wordings.

Condorcet writes [19] p. 126: “Create an opinion of those  $N(N-1)/2$  propositions that win most of the votes. If this opinion is one of the  $N!$  possible then consider as elected that subject to which this opinion agrees with its preference. If this opinion is one

of the  $(2^{\binom{N-1}{2}}) - (N!)$  impossible opinions then eliminate of this impossible opinion successively those propositions that have a smaller plurality and accept the resulting opinion of the remaining propositions.”

In short, Condorcet suggests that the weakest pairwise defeats should be eliminated successively until the remaining pairwise defeats form a ranking of the candidates. The problem with Condorcet’s proposal is that it is not quite clear what it means to “eliminate” a pairwise defeat (especially in so far as when one successively eliminates the weakest pairwise defeat that is in a directed cycle of not yet eliminated pairwise defeats until there are no directed cycles of non-eliminated pairwise defeats any more then the remaining pairwise defeats usually do not complete to a unique ranking [20]). It is clear what it means when a candidate is “eliminated”; this candidate is treated as if he has never stood. But what does it mean when the pairwise defeat  $A > B$  is “eliminated” although candidate A and candidate B are still potential winners?

A possible interpretation would be to say that the “elimination” of a pairwise defeat is its replacing by a pairwise tie. However, when this interpretation is being used then the Smith set, as defined in the Introduction, can only grow but not shrink at each stage. But when the Schwartz set, as defined in the Introduction, is being used, then the number of candidates decreases continuously. With the concept of the Schwartz set the Schulze method can be described in a very concise manner:

**Step 1:** Calculate the Schwartz set and eliminate all those candidates who are not in the Schwartz set. Eliminated candidates stay eliminated.

If there is still more than one candidate and there are still pairwise comparisons between non-eliminated candidates that are not pairwise ties: Go to Step 2.

If there is still more than one candidate, but all pairwise comparisons between non-eliminated candidates are pairwise ties, then all remaining candidates are *potential winners*: Go to Step 3.

If there is only one candidate, then this candidate is the *unique winner*.

**Step 2:** The weakest pairwise defeat between two non-eliminated candidates is replaced by a pairwise tie. Pairwise comparisons that have been replaced by pairwise ties stay replaced by pairwise ties.

In the version in section 4, the *weakest pairwise defeat* is that defeat where  $|d[i,j] - d[j,i]|$  is minimal.

In the version in appendix C, the *weakest pairwise defeat* is that defeat where the number of votes for the winner of this pairwise defeat is minimal or —if there is more than one pairwise defeat where the number of votes for the winner is minimal— of all those pairwise defeats where the number of votes for the winner is minimal that pairwise defeat where the number of votes for the loser of this pairwise defeat is maximal.

If the weakest pairwise defeat between non-eliminated candidates is not unique, then all weakest pairwise defeats between non-eliminated candidates are replaced by pairwise ties simultaneously. Go to Step 1.

**Step 3:** The TBRC is calculated as described in section 2. The winner is that potential winner who is ranked highest in this TBRC.

## E An Example without a Unique Winner

Example [21], p. 502:

3	ABCD
2	DABC
2	DBCA
2	CBDA

The matrix  $d[i,j]$  of pairwise defeats looks as follows:

	A	B	C	D
A	—	5	5	3
B	4	—	7	5
C	4	2	—	5
D	6	4	4	—

The matrix  $p[i,j]$  of the path strengths looks as follows:

	A	B	C	D
A	—	1	1	1
B	1	—	5	1
C	1	1	—	1
D	3	1	1	—

Candidate X is a potential winner if and only if  $p[X,Y] \geq p[Y,X]$  for every other candidate Y. Therefore, candidate B and candidate D are potential winners.

When the Schwartz set heuristic is being used then at the first stage the Schwartz set is calculated. The pairwise defeats are  $A > B$ ,  $A > C$ ,  $B > C$ ,  $B > D$ ,  $C > D$ , and  $D > A$ . Hence, the Schwartz set is: A, B, C, and D. At the second stage, the weakest pairwise defeat that is not a pairwise tie between candidates who have not yet been eliminated is replaced by a pairwise

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tie. The weakest pairwise defeats are  $A > B$ ,  $A > C$ ,  $B > D$ , and  $C > D$  each with a strength of 5:4. All these pairwise defeats are replaced by pairwise ties simultaneously. The remaining pairwise defeats are  $B > C$  and  $D > A$ . Hence, the new Schwartz set is: B and D. Since there are now no pairwise defeats between candidates who have not yet been eliminated, the algorithm stops and candidate B and candidate D are the winners.

Since 5 voters strictly prefer candidate B to candidate D and 4 voters strictly prefer candidate D to candidate B, candidate B is ranked higher than candidate D in the TBRC with a probability of 5/9 and candidate D is ranked higher than candidate B in the TBRC with a probability of 4/9. Therefore, the winner of the Schulze method is candidate B with a probability of 5/9 and candidate D with a probability of 4/9.