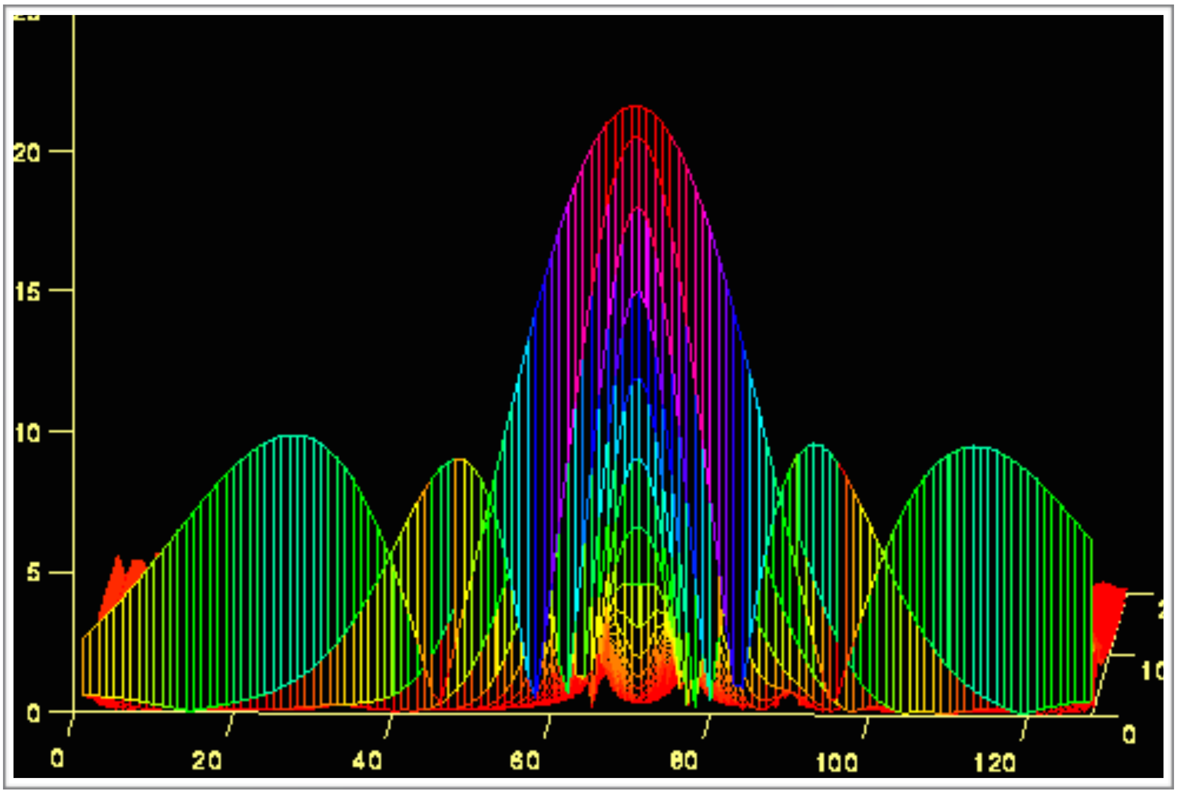
Discrete Wavelet Transform



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Introduction

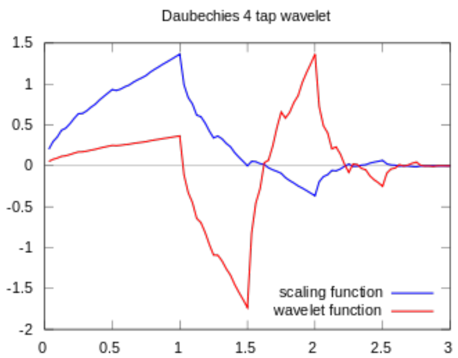
Wavelets break signals up and then analyse them separately with resolution that is matched with scale. Wavelets have been a very important discovery in the signal processing/analysing industry. They have been very useful in compressing signals, removing noise out of signals, edge detection and in earthquake prediction. The history behinds wavelets is quite interesting as many a times research was left on standby and many sparks were needed to lead to its invention. Basics behind wavelets and their characteristics will need to be covered to understand their use in complex applications. Also one of the transforms using wavelets will be described in detail with an application of that transform. [1]

History

The concept of wavelet analysis is a fairly new method. However, he idea of studying signals by representing them as functions isn't new. In the 1800s Joseph Fourier found out that he could superpose (overlap) sines and cosines to represent other functions. This opened a new world of looking at functions. The first time the term “wavelets” was heard was in A. Haar thesis in 1909. In the 1930s many independent groups worked on basis functions. A research by Littlewood, Paley and Stein discovered a function which could vary in scale and conserve energy when analysing function energy. This helped David Marr in making a algorithm for image processing using wavelets in the 1980s. [1]

Guido Weiss and Ronald Coifman were finding a common function which could reconstruct all elements of the function space using atoms. Grossman and Morlet found wavelets in quantum physics. Then Stephane Mallat in 1985 included the concept of wavelets in his work on digital signal processing which gave wavelets strength and importance. [1]

A few years later is were the most important part of wavelet research was done. Ingrid Daubechies constructed orthonormal basis functions. In a 3 demential plane every vector is combination of these basis vectors a=(1,0,0),b=(0,1,0),c=(0,0,1). Orthonormal means that those vectors are perpendicular to each other which makes calculations easier. Similarly there are basis functions that can manipulated to construct a signal. The coefficients that are needed to define a signal were made easier to find because of Daubechies's work. This work has probably become the "cornerstone in wavelet applications". [1]



Daubechies wavelet [10]

Joseph Fourier [12]

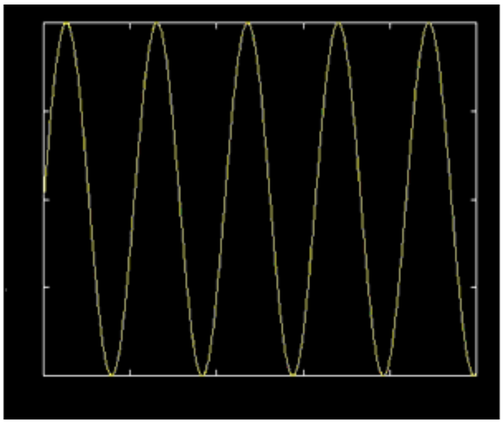
Ingrid Daubechies [15]

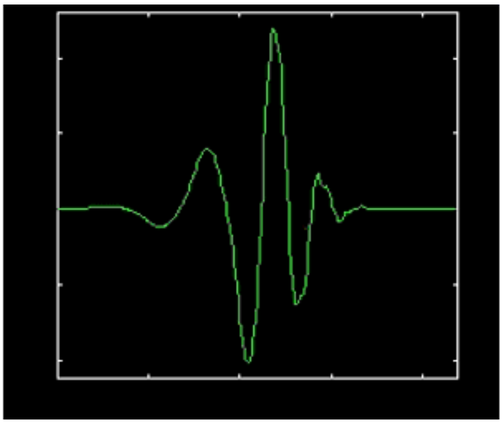
Ronald Coifman [14]

Basics

Definition

A wavelet is a oscillation in which the amplitude starts at zero. It increases to a certain point and decreases back to zero again. Main idea behind the wavelet is to study a signal by its scale. The Fourier analysis breaks the signal into sine waves of different frequencies. On the other hand a signal is broken into various wavelets which are transformations of the "mother wavelet" in wavelet analysis. [2]

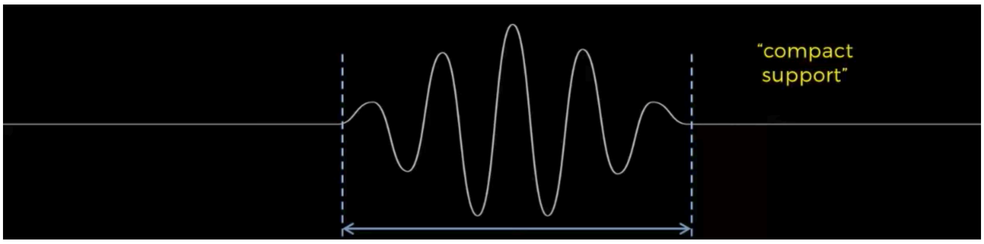




Sin wave (left) compared to the Daubechies 5 wavelet (right)

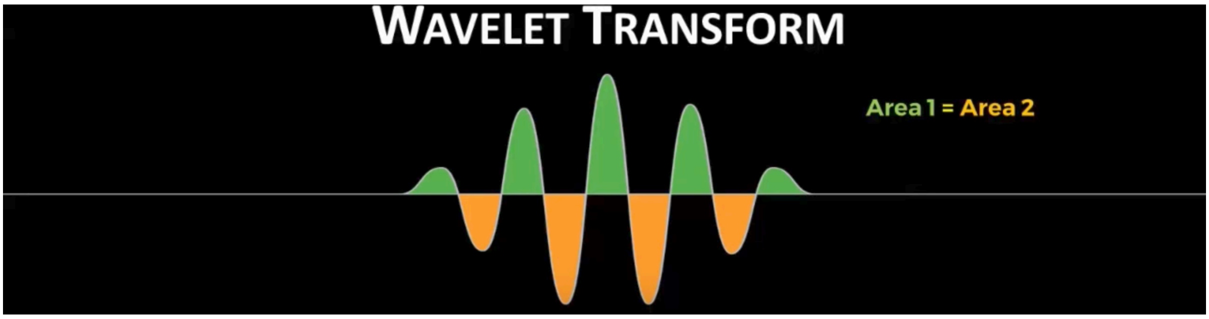
Characteristics

In the figure above one can see that the sin wave is smooth and has infinite length but the wavelet is irregular/jagged and is finite (compact). These features of the wavelet make them ideal for signal analysis. The jagged edge can help them analyse signals with discontinuity and sudden changes.The compact support means that the wavelets vanish when outside a finite interval. This helps them to localise in time and frequency whereas the Fourier transform is only localised in frequency. So with wavelets one knows where a particular part of the frequency domain occurs in time but in Fourier transform the frequency domain cannot be associated with time .[2]



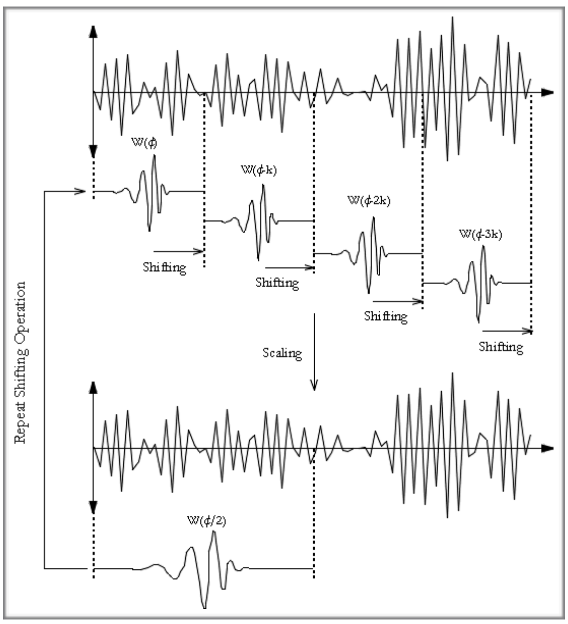
Wavelet is compact and oscillations aren't infinite [4]

The area under the curve should be zero in the case of wavelets so the energy is distributed the same in the negative and positive direction. [4]



Area is equal for equal energy distribution [4]

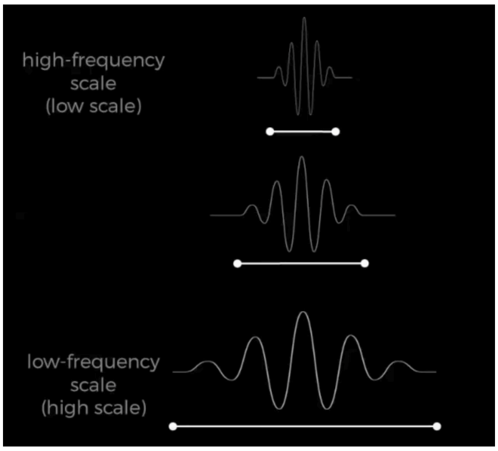
The wavelet can localise in the time domain because of the translations of the "mother wavelet" and in the scale (frequency) via the dilations. These transformations are used to calculate the wavelet coefficients. These coefficients show the correlation between the wavelet and localised signal. Each wavelet segment has a coefficient which gives a time scale function and relates the wavelet to the signal. [2]



Translation and scaling of mother wave [2]

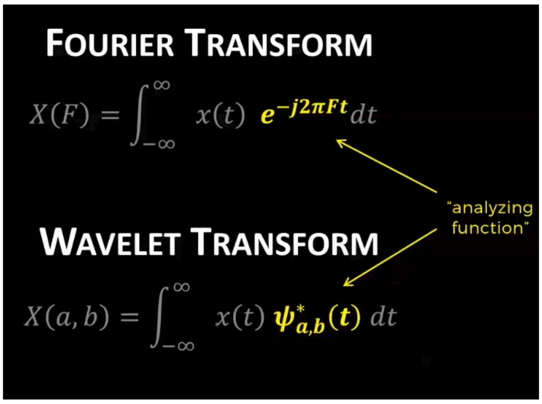
This diagram above shows how the mother wave is either translated or scaled to correlate to certain parts of the signal. [2]

The wavelet is compressed for higher frequency and stretched for lower frequency. Stretching a wavelet increases its length and makes it less accurate at finding the time at which a particular low frequency occurs. The more compressed the wavelet the lower the scale it and the more stretched the wavelet the higher the scale.[4]



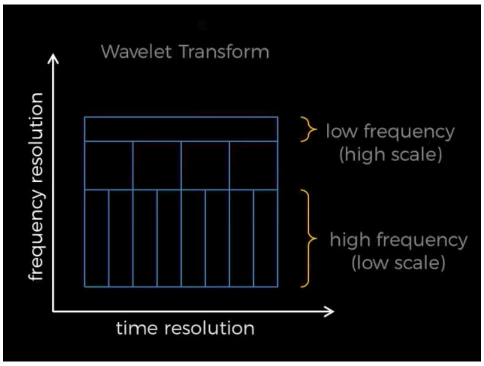
Differently scaled wavelets [4]

In the Fourier Transform we multiply a signal which is function of time with a analysing function(in figure below). In the wavelet transform we multiply by a wavelet analysing function. The output of the Fourier transform is coefficients which correlate to only the frequency (X(F) in the diagram). On the other hand the wavelet transform outputs a 2*2 matrix which is comprised of their scale and translation(X(a , b)in the diagram). [4]



Difference in analysing function[4]

Heisenberg's uncertainty principle says that one cannot be extremely accurate in location in time and location in frequency. [4]



Wavelet transform Heisenberg boxes[4]

These are are Heisenberg boxes. For the high frequency parts of the signal one can achieve great resolution in time as high frequency parts have compact support (meaning they are compressed and wavelets can easily find detail as they are compact as well). But in this case one doesn't have great resolution in frequency (the number of boxes represent resolution). The Heisenberg boxes must have the same area in low and high frequency. In this case we have less resolution in time (4 boxes compared to 8 in high frequency) the signal is spread apart and does not offer compact support. However, we have better resolution in frequency in the low frequency (2 boxes compared to 1 box in high frequency).[4]

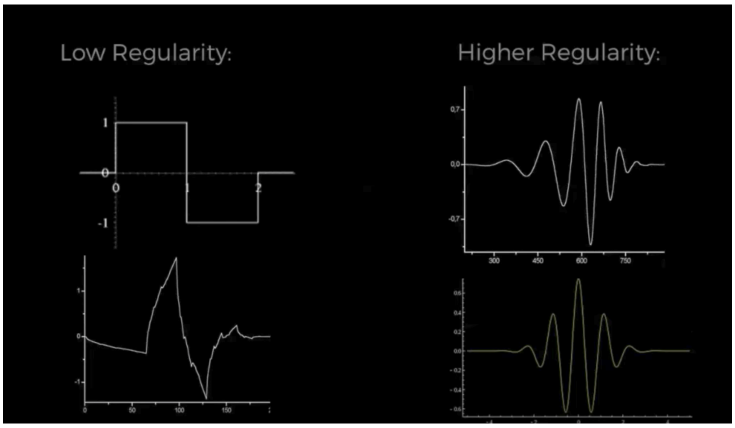
Vanishing Moments

A key way to differentiate between wavelets is to see how many vanishing moments a wavelet has. A vanishing moment is defined as follows. "So the kth moment of a function f is defined by the integral of the function multiplied to the variable of the function raised to k and the kth moment vanishes if this integral is zero."[4]

$$m_k = \int_{-\infty}^{\infty} f(x)\, x^k \, dx$$

Moment k formula[4]

The more vanishing moments a wavelet has the more complex it is and the more accurately it can represent a signal. However, having more vanishing moments means that there is longer support rather than compact support. Low regularity will lead to jagged representations of the signal where as high regularity will lead to smooth representation of the signal. [4]



Different Regularity [4]

The more vanishing moments a wavelet has the greater regularity it has as more vanishing moments lead to a more accurate representation. [4]

Discrete Wavelet Transform(DWT)

Discrete wavelet transform helps to show many natural signals with fewer coefficients making it very useful for signal compression and signal demonising to eliminate redundancy and lower the memory required to store the signal.[5]

In DWT one ends up with the same number of coefficients as the original signal but many of those coefficients are close to zero and hence can be removed without much loss in quality. The important parts of a signal are captured by some large magnitude coefficients and noise and disruptions are made of small coefficients which can be discarded. When inverted the DWT can give a perfect reconstruction of the original signal.[6]

The main difference between DWT and CWT(continuous wavelet transform) is that CWT discretises the scale more finely. In DWT the scale parameter is discretised by integer power of 2, 2^j where j=1,2,3…., whereas in CWT one would have fractional powers of 2 so more sampling and hence more computational time. In the DWT the translation is always proportional to the scale, so the formula becomes $2^j m$, where m is nonnegative integer. This is the discretised wavelet formula below.[6]

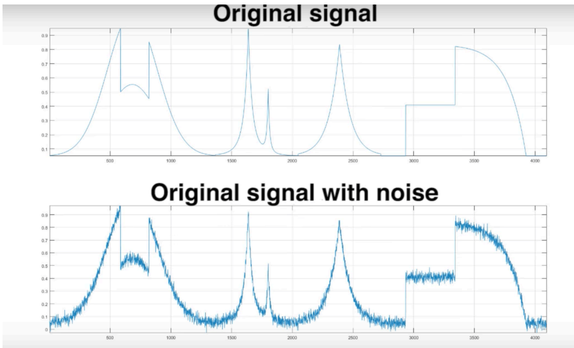
$$\boxed{\frac{1}{\sqrt{2^j}}\,\psi\Big(\frac{1}{2^j}(n - 2^j m)\Big).}$$

Discretised Wavelet Formula[6]

A wavelet can be constructed using a scaling function which describes the scaling properties of that particular wavelet.[7]

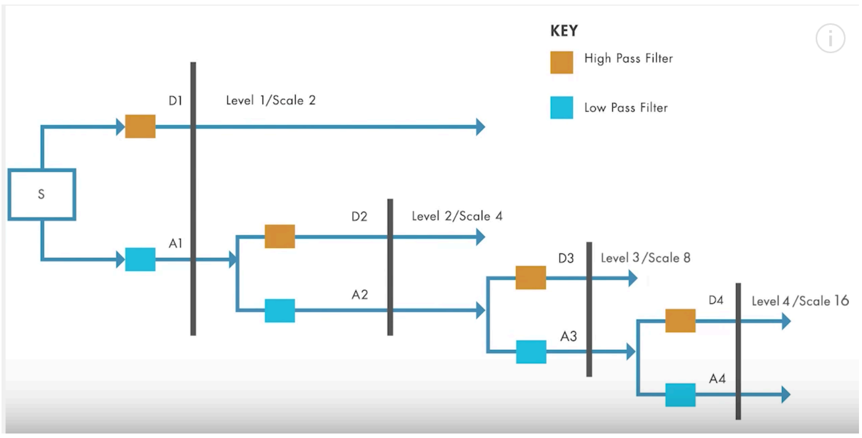
The main applications of the discrete wavelet transform is data compression and signal demonising. DWT can compress data as it outputs few high magnitude coefficients which reduce memory and also they can remove the redundancy in signals which is known as noise by eliminating the lower magnitude coefficients which disrupt the signal.[7]

<u>De noising a signal using DWT</u>



We will de noise a noisy signal (signal with disruptions and unwanted data) using the discrete wavelet transform. White noise is not correlated in time and it effects every frequency component and is hence the hardest to remove. [8][9]

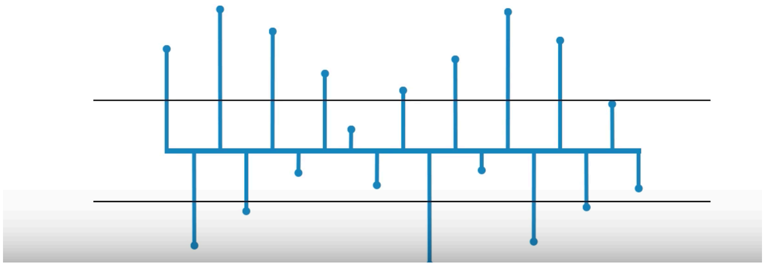
First we will try to get the detailed coefficients and terminate the reductant parts using a multilevel wavelet decomposition. [8]



Multilevel wavelet decomposition[8]

In the diagram above one can see that the signal is passed through a high pass and low pass filter. The high pass filter is known as the detail level whereas the low pass filter is known as the approximation level. The approximation band can be decomposed several times to get a finer level. The D1 captures the high frequency part of the signal which is mainly the noise. However, some of this high frequency is because of abrupt changes in the signal which carry meaning. To obtain these abrupt changes and eliminate noise one would need to scale by a threshold. The thresholding technique used here would be the universal threshold. [8]

There are two thresholding operations one is soft thresholding and the other is hard thresholding. [8]



Detailed coefficients [8]

In both cases the coefficients less than the threshold (black line) are set to zero as they are considered redundant. In soft thresholding the coefficients greater than the threshold are reduced towards zero by subtracting the value of the threshold from the original values. However, in hard thresholding the coefficients greater than the threshold are left unchanged. [8]

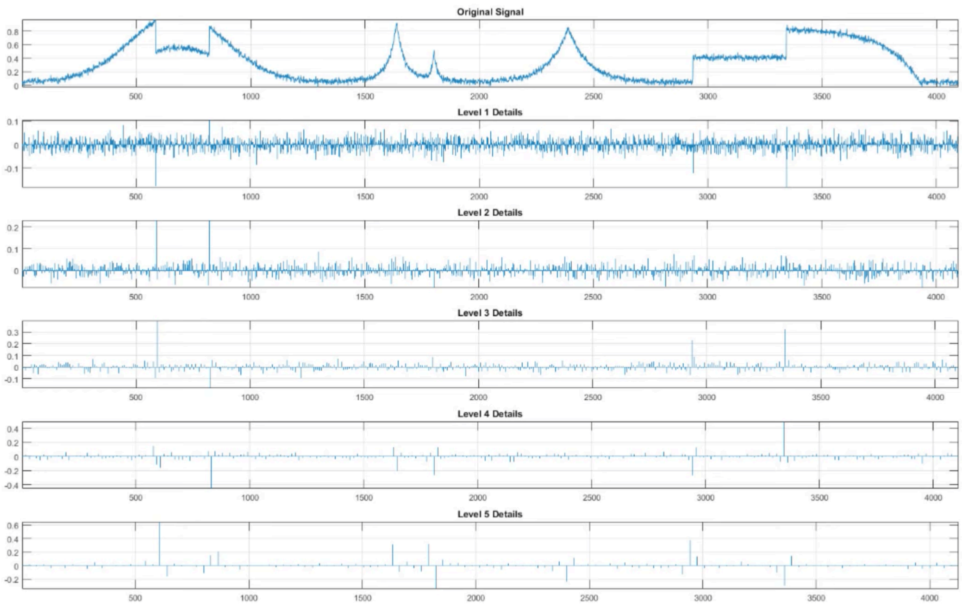
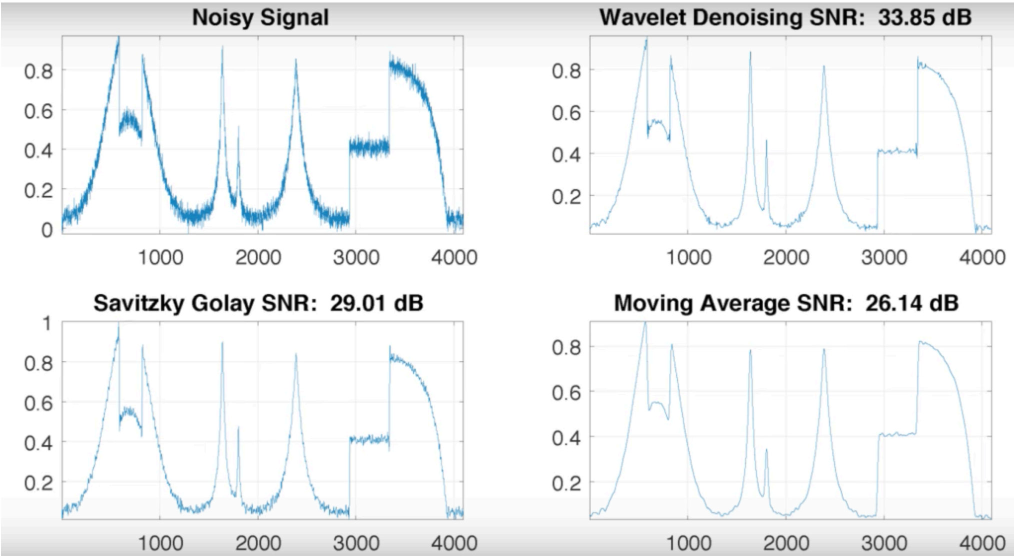


Image of the coefficients after each level[8]

This image shows us how the number of coefficients decrease as the scale increases this removes small magnitude coefficients and keeps the high magnitude ones. After thresholding the signal is reconstructed.



Wavelet transform compared to other techniques[8]

As compared to other methods the DWT gives the best result. Compared to the Golay the wavelet transform removes more noise and and provides the most accurate representation of the original signal.

Conclusion

Wavelets were invented in over one hundred years. They are especially genius as they are compact and can locally analyse signals. The DWT is a very effective method to de noise signals and saves a lot of memory. Special characteristics like these see wavelets appearing in many various applications and research on them is advancing with great pace.

Citations

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