Math of the universe
Paper 1
Sequences
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2017/07/17

Sequences

Introduction

Have you ever stamped your foot while listening to music? Have you ever counted like 1, 2, 3, 4 while you are doing a dance or movement? Have you ever group things into different numbers? Surely, the answer is definite. Everyone has these experienced; and it turns out that throughout the daily lives, we have created sequences without realizing. They are not sticked with calculations and problem solving in schools. They can actually be so simple that they are everywhere. And the wide applications behind this simplicity grants it the charm that fascinated people. Whether it is to figure out the number of rabbit offsprings; calculate the area of an irregular shape formed by a curve line and a straight line; or the combinations of possible paths on staircase-squares. These all sound like algebra or calculus problems, but actually, sequences are good at representing these situations probably even better than formulas sometimes. These applicable sequences have changed people's minds and ways of treating regular situations. It is such a path towards the beauty of lives.

In this paper, general facts about sequences and some interesting sequences will be discussed and presented with addition to the applications and creative perspectives of discovering patterns of a sequence. Major types of sequences being discussed will be Arithmetic and Geometric sequences. These will be introductory and foundational to the latter content. Afterwards, special sequences such as Fibonacci, Catalan, Triangular and Square numbers and patterns in the Pascal Triangle will be presented.

Arithmetic and Geometric Sequences

Before going into details about specific sequences, it is important that basic facts about sequences are understood. Arithmetic sequence and Geometric sequence are the two general types of sequences. They divided sequences into two groups, one associating with addition being the Arithmetic sequence, the other associating with multiplication being the Geometric sequence.

Definitions

Accurately saying, the definition for arithmetic sequence is that there is a common difference between each term in the sequence, in other words, a term minus the term before it should always result in the same value. For example:

"1,2,3,4,5,6,7,8,9,10……"

This is a sequence with consecutive integers with an addition of 1 each time. Since the differences are the same, it is a arithmetic sequence. Take another sequence as example:

"100, 98, 96, 94, 92, 90……"

This is also an arithmetic sequence, but in a descending trend. It has a difference of -2. An arithmetic sequence does not always have to have a positive difference as what people also think of it.

The definition for geometric sequence is that there is a common ratio between two consecutive terms, meaning that each term is being multiplied by the same number to get the next term. For example:

"2,4,8,16,32,64,128,256……"

In this sequence, each term is being multiplied by 2 to get the next term, so it is obviously a geometric sequence. Another example might be:

"1, -1, 1, -1, 1, -1, 1, -1……"

This is not as obvious as the previous example, but it is still a geometric sequence since it has a common ratio of -1. Therefore, it is very important to not be deceived by the appearances of sequences. [1]

<u>Applications</u>

Due to their properties, arithmetic and geometric sequences are very capable of representing some daily life situations or problems. Arithmetic sequences can represent situations when constant changes are being added to an object or a scenario. One of them is simple interest rate, which is highly applied in our daily lives. Say as a customer wants to buy an insurance with base price 100 dollars and pay 10 dollars every week. This can create the following sequence:

"100, 110, 120, 130, 140, 150……"

A constant addition of 10 is being added to the first term, which is 100, and in this way, the money accumulates and each term represents the customer's total payment in the nth week. This is a typical use of arithmetic sequences in our daily life, we use it more often than we realize.

Geometric sequences have more usages compared to arithmetic sequences. Its general property, having a common ratio, makes it a well representation of insurance cases, bank interest rate, or similar situations. A typical example will be compound interest rate. Say as the bank wants to know how much money each year will Customer A have if she deposit 10,000 dollars at the bank and the interest rate is 1% per year. Then the sequence to represent this situation will be like the following:

"10000, 11000, 12100, 133310, 14641……"

Each term of the sequence represents the money owned by Consumer A in the bank account from year 0 to year infinity. This is the most common application; there is a more creative use of the geometric sequence. [2]

<u>Archimedes Theorem</u>

Some geometric sequences have the property that they have infinite terms but a finite sum. Utilizing this property, Archimedes came up with the Archimedes Theorem which is a “dissection of a parabolic segment into infinitely many triangles.”[footnote] This theorem helps to compute the area enclosed by a parabola and a straight line. This theorem is an use of the sum of a geometric sequence.

(See Figure 1) To begin with, Archimedes first dissect the area into infinite amount of triangles and assumes that the blue area is equal to 1 and calculated that the green area is exactly 1/4 of the blue area, the yellow area is exactly 1/4 of the green area and it goes forever like the following geometric sequence:

“1, 1/4, 1/16, 1/64……”

The sum of the entire area is, therefore, all the triangle areas adding together. By using the equation for the sum of an infinite geometric sequence, which is:

$$\sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{1-r}$$

Archimedes get the answer 4/3, which is proven to be the correct answer by other ways. This is such a creative way of using geometric sequences that no one had thought of it before Archimedes, and was later named the Archimedes theorem. [2]

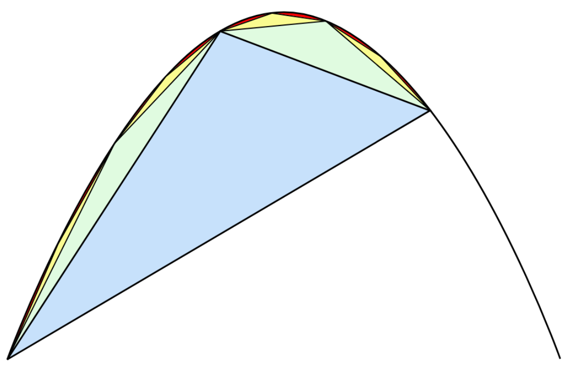

Figure 1: Archimedes Theorem

Further Investigation

Furthermore, arithmetic sequences and geometric sequences sometimes hide inside a sequence. There are two special ones. The first one being a geometric sequence hiding in the differences of a sequence. Take the below sequence as an example:

"1, 2, 5, 14, 41……"

These numbers seem to be neither arithmetic nor geometric sequence; but actually, if we take the second term minus the first one, the third term minus the second one and so on, we will get a geometric sequence like the following:

"1, 3, 9, 27……"

This is definitely a geometric sequence since it has a common ratio of 3. Therefore, the original sequence does have a pattern. The general term formula of this sequence is also related to the geometric sequence. In fact, the general term formula of this sequence is a transformation from the summation formula of geometric sequences. The formula is presented as:

$$General\ Term\ Formula = \mathrm{A(n)} = a_1 + b_1 \times \frac{1(1-r^{n-1})}{1-r}$$

"A1" is the first term in the original sequence and "B1" is the first term in the geometric sequence. Additionally, "r" is the common ratio in the geometric sequence. The last part of the formula, which is the fraction part, is very much the same as the summation formula of geometric sequences.

The second special sequence is a arithmetic sequence hiding in the ratios of a sequence. As an illustration:

"1, 1, 2, 6, 24, 120, 720……"

The pattern in this sequence is also not obvious, but if we divide the second term by the first term do this for the rest, we will be able to see the pattern. Like the following:

"1, 2, 3, 4, 5, 6……"

It is, indeed, an arithmetic sequence. These are some special cases when steps have to be done before the pattern appears. Sometimes, it might be 3 or more steps. The general term formula of this type of sequence with an arithmetic sequence of difference 1 is presented as the following:

$$General\ Term\ Formula = A(n) = (n-1)!$$

Finally, it is interesting that we can find a geometric sequence in the differences of a sequence and an arithmetic sequence in the multiples of a sequence. This, again, shows different angles of treating sequences and problems.

Interesting Sequences

Fibonacci Sequence

Fibonacci sequence seems to be the best well-known sequence for everyone. Most people think that it is cool to know the Fibonacci numbers just because everyone else does; but there are many interesting properties that make Fibonacci Sequence the very most famous sequence.

Starting from when it was presented to the world by Fibonacci, more people begin to find Fibonacci numbers in plants, spirals, natures and structures etc. These numbers fascinated scientists and mathematicians for centuries, and the sequence still has its charm.

Figuratively speaking, Fibonacci Sequence is obtained by adding 2 previous terms to get the next term. It is also given that the first and second term are equal to 1. This is the Fibonacci Sequence:

"1, 1, 2, 3, 5, 8, 13, 21……"

We can see that one plus one equals two, therefore the third term is 2; and one plus two equals three which is the fourth term. This pattern continues for an infinite length; and the sequence contains much more complexity than it appears to be.

It is discovered that Fibonacci sequence is closely related to the Golden ratio. Golden ratio is highly applicable in buildings, structures, photographs, visuals and sceneries etc. It is equal to approximately 1.618034. This number is found in the Fibonacci sequence. As the sequence approaches infinity, a term (An) in the Fibonacci sequence divide by the previous term (An-1) results in a closer value to the golden ratio. For instance, 3 divides by 2 equals 1.5; 13 divides by 8 equals 1.625; 21 divides by 13 equals 1.615. It is obvious that the value get closer and closer to the golden ratio. This can be proved by geometry as well. If we divide the golden rectangle, which is a rectangle with length divides by width equaling golden ratio, into smaller rectangles according to the Fibonacci numbers; we can clearly see that they fit perfectly. *(See Figure 2)* Also, as the length and width accumulate to bigger values as the Fibonacci sequence approaches infinity, it is closer to fitting the golden rectangle. Since the length of the rectangle divides by the width is equal to the golden ratio, as Fibonacci numbers fit the rectangle in a better sense, it is going to be closer to golden ratio as well. This shows the connection between Fibonacci numbers and the golden ratio. [3]

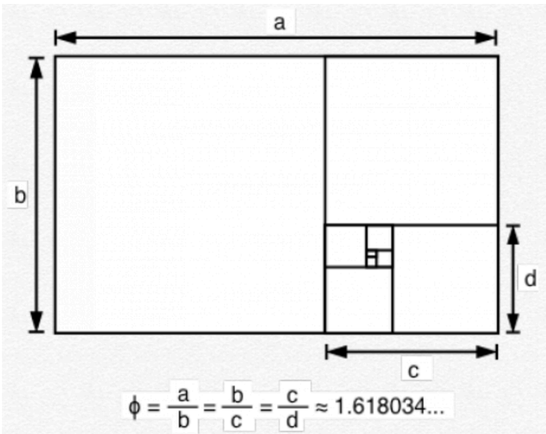

Figure 2: Golden Triangle

The Fibonacci Sequence also becomes interesting when we take the numbers and square them. The square of the Fibonacci numbers becomes the sequence below:

"1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025……"

When we add two consecutive Fibonacci numbers, we get the next Fibonacci number and this is the pattern. A pattern, surprisingly, also appears when we add two consecutive squares of Fibonacci numbers.

Fibonacci numbers: "1, 1, 2, 3, 5, 8, 13, 21……"

Squares of Fibonacci numbers: "1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025……"

Adding two consecutive squares of Fibonacci numbers: "2, 5, 13, 34, 89……"

By doing the above simple calculations, we see that when we add two consecutive squares of Fibonacci numbers, we will get Fibonacci numbers again. These numbers appear in the Fibonacci sequence with exactly an interval of 1 number. This is, however, not the only pattern. When we sum the squares of Fibonacci numbers, we also get a pattern:

Sum of squares of Fibonacci numbers:
1+1=2
1+1+4=6
1+1+4+9=15
1+1+4+9+25=40
……

We will get 2, 6, 15, 40 as our results when we add up the first five terms step by step. These values are not Fibonacci numbers and seem to be usual. However, if we go back to the Fibonacci numbers and multiply them, we will see the pattern.

Multiplying two consecutive Fibonacci numbers:
1x2=2
2x3=6
3x5=15
5x8=40
……

These products of two consecutive Fibonacci numbers pair up perfectly with the sums of squares of Fibonacci numbers. As this pattern was discovered, mathematicians were amazed and they tried to find the reason behind this pattern. Consequently, it was found that geometry can explain this pattern.

If we take the Fibonacci numbers and create squares that have lengths according to these numbers, they will form a rectangle like Figure 3. In order to prove this pattern, we just simply has to calculate the area of the rectangle in two different ways. In the first method, we calculate the area of the rectangle by adding the smaller triangles together which turns out to be the sum of the squares of the Fibonacci numbers. This is because the area of a square is computed by the square of the length and since the lengths are equal to Fibonacci numbers, the areas will be equal to the squares of Fibonacci numbers. The second way of calculating the area of the rectangle is to multiply the length with the width like what we learnt in elementary schools. The length happens to be the largest Fibonacci number applied plus the previous Fibonacci number and the width is just the largest Fibonacci number applied. As illustration, in the figure, 8 is the length of the largest square, the largest Fibonacci number, so it is the width; 5 plus 8 is then the length. At last, the length, 13, multiplies the width, 8 is equal to 104. 104 is also equal to the sum of areas of the squares together. This is the reason why the sum of the squares of Fibonacci numbers is equal to the product of two consecutive Fibonacci number. [4]

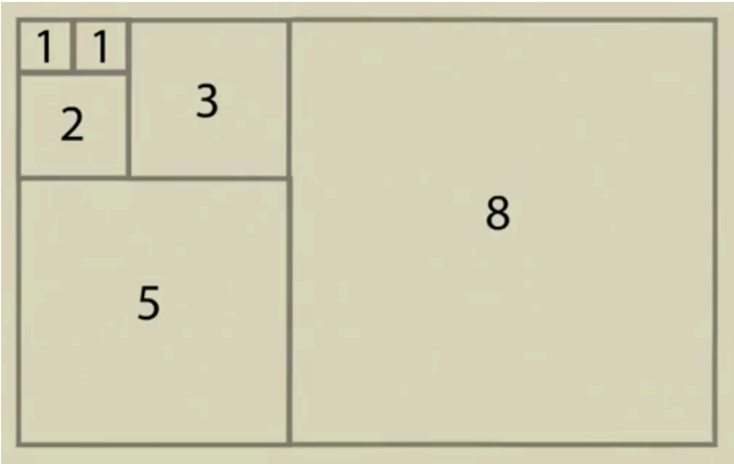

Figure 3: Rectangle forms by Fibonacci numbers

Catalan Sequence

Exclude the Fibonacci sequence, there are some other interesting sequences as well; one of them is the Catalan Sequence. The Catalan sequence is a more complicated sequence. It was discovered when Euler was dealing with triangulations of polygons. Later, he found out that the number of ways of triangulating polygons can form a number sequence. This sequence was perfected by Catalan who it named after.

The sequence is presented as:

"1, 1, 2, 5, 14, 42, 132, 429, 1430……"

And the general term formula of this sequence is presented as:

$$A(n) = \frac{(2n)!}{n!(n+1)!}$$

The Catalan Sequence is related to many mathematical problems. One of them being the triangulation of polygons, the very first discovered. This is a problem associating with dividing a polygon into triangles by drawing lines from one angle to the other and without crossing. It was discovered that the number of ways of doing these triangulations to a n+2 polygon (every shape has to have length bigger than 2) can be represented by Catalan numbers. For instance, the number of way to triangulate a triangle is 1, for square there are 2 ways, 5 ways for pentagon and 14 ways for hexagon. *(See Figure 4)*

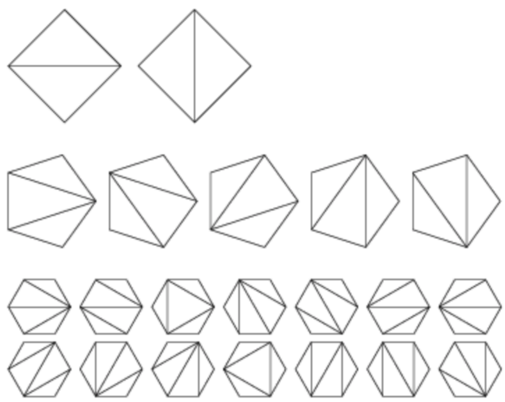

Figure 4: Catalan numbers

It can also serve as a representation for a math problem that ask for the number of ways of getting from one end of the diagonal of a grid to the other other with only step to the right and step upwards. *(See Figure 5)* For example, in order to get to the point (1, 1), there is only one way, which is to go right and then upwards. In addition, there are 2 ways going to the point (2, 2) and 5 ways going to the point (3, 3) etc. After computing for all the possible ways of reaching each coordinate on the grid, we can see that the longest diagonal presents a Catalan sequence. This shows how the Catalan sequence can be used to solve problems like this. [5]

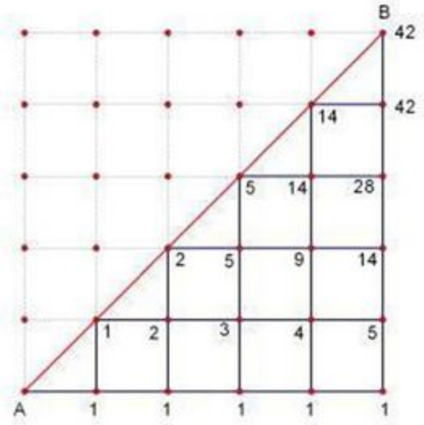

Figure 5: "Grid problem"

Polygon numbers (Triangle, Square Pentagon numbers)

Like the Catalan numbers, polygon numbers are found in geometry or shapes as well. In this paper, I will only discuss triangle numbers, square numbers and pentagonal numbers as example sequences.

To begin with, triangle numbers are created by enlarging a triangle starting with a dot, like shown in Figure 6. Then the triangular numbers are discovered to be:

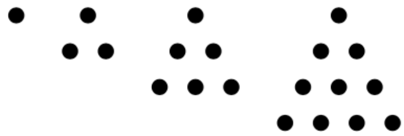
Figure 6: Triangular numbers

"1, 3, 6, 10, 15……"

These are all the sums of of dots in the nth triangle. These sums can be calculated by an arithmetic sequence and the summation formula of arithmetic sequence, like the following:

"1, 2, 3, 4, 5, 6, 7, 8……"

$$Summation\ Formula = S(n) = \frac{a1+an}{2} \times n$$

This arithmetic sequence is the number of dots on each horizontal line of the triangle and by adding them, we get a triangular number. For instance, in order to get the second term of the triangular number sequence, we need to add the first two term of the arithmetic sequence together. By using the summation formula, we get 3 as a result, which is proven to be correct.

Not only triangular numbers, but square numbers can also be calculated through the same method. Square numbers are created by using dots to present the n by n length squares, starting from 1 by 1. *(See Figure 7)* The sequence is presented as the following:

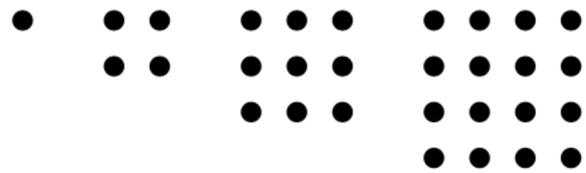

Figure 7: Square numbers

"1, 4, 9, 16……"

These are sums of dots in the nth square. The arithmetic sequence for calculating square numbers is less obvious; one need to divide the dots in the square as shown in Figure 8 to see the hidden sequence. By counting the number of dots connected by each of the lines, we get the following arithmetic sequence:

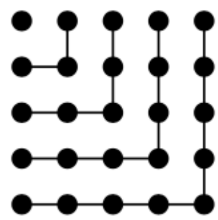

Figure 8: Square numbers

"1, 3, 5, 7, 9……"

This arithmetic sequence has a difference of 2. The arithmetic sequence summation formula can also help to calculate the triangular numbers. If we want to get the nth square number, we just simply have to add the first nth numbers in the arithmetic sequence together, just as what have been done for the triangular numbers.

Finally, there is also the pentagonal numbers. These numbers are formed in basically the same way as triangle numbers. The pentagon becomes larger by adding another

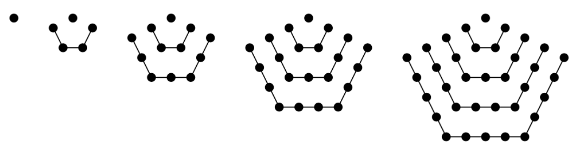

Figure 9: Pentagonal numbers

layer and the summations of dot in the pentagon forms the pentagonal numbers. These numbers are presented as:

"1, 5, 12, 22, 35……"

And Figure 9 shows the sequence in geometry shapes. A similar arithmetic sequence to the ones for triangular and square numbers can be used for calculating this sequence as well. It is presented as the following:

"1, 4, 7, 10, 13, 16……"

The difference between every term is 3 and hence it is an arithmetic sequence. Each of the number, if we look back at Figure 9, is the number of dots being added to the pentagon each time. Therefore, it is not hard to understand that by adding the first nth term of this arithmetic sequence together, we will get the nth pentagonal number. Furthermore, if one looks closer at the three arithmetic sequences utilized to compute for triangular, square and pentagonal numbers, a pattern appears as well. The common difference in the arithmetic sequence for calculating triangular numbers is 1, for square numbers is 2 and 3 for pentagonal numbers. The difference increases by 1 when the number of sides increases by 1. They are directly proportional.

Exclude finding pentagonal numbers from the drawings and the arithmetic sequence, there is also a more convenient way which is through an explicit formula. The explicit formula for finding a general term of the pentagonal numbers is:

$$\frac{k(3k-1)}{2}$$

This is the fastest and easiest way to find a pentagonal number. The only thing that needed to be done is to plug in "k", which is the position of the pentagonal number like 2 for second and 3 for third. [6]

Pascal's Triangle

Last but not least, the Pascal's Triangle (See Figure) also hides many interesting sequences inside it; Fibonacci numbers and triangular numbers are all part of it.

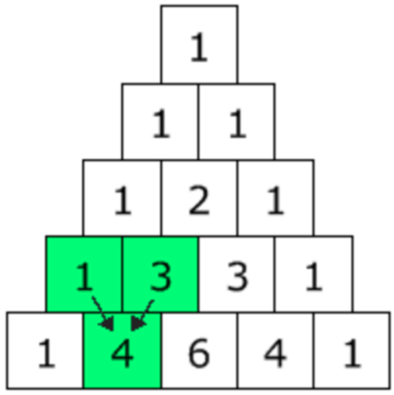

Figure 10: Pascal's triangle

Pascal's Triangle is created by having the two edges of a pyramid being ones and the rest of the numbers forming through adding the two numbers above them. *(See Figure 10)* This triangle is found to have close relation with many sequences and has important applications.

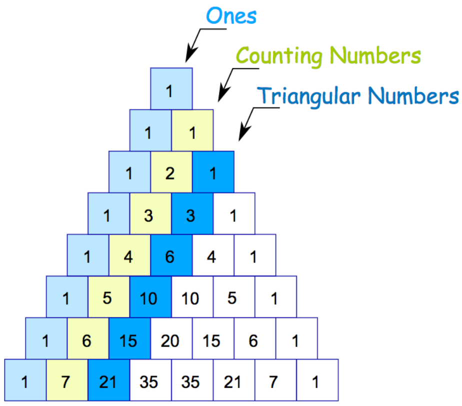

Figure 11: Triangular numbers in Pascal's triangle

Firstly, it contains triangular numbers which are on the third diagonals of the triangles. The pattern shows in Figure 11 continues as the Pascal's Triangle grows. Secondly, if we add each line of the Pascal's triangle, we can also find a pattern. The first sum will be 1; then the second one will be 2; the third one is 4; after that is 8. (See Figure 12) As a result, one can see that they double each time, meaning that they are all powers of 2. Thirdly, by simply looking at each line on the Pascal's triangle, a pattern is presented. Unexpectedly, these numbers are all powers of 11. 1 is 11 to the power of 0; 11 is 11 to the power of 1; 121 is 11 to the power of 2; 1331 is 11 to the power of 3 etc. This pattern continues as shown in Figure 13. Fourthly, for numbers on

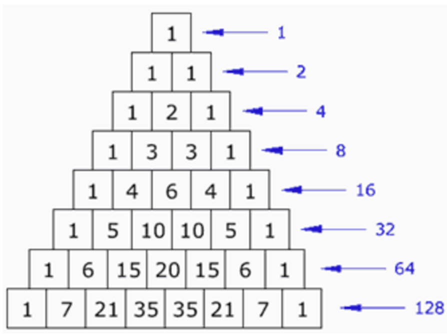

Figure 12: powers of 2

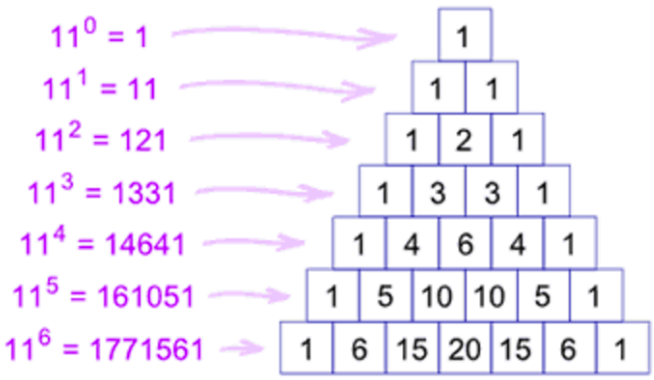

Figure 13: powers of 11

the second diagonal of Pascal's triangle, their squares can also be resulted by adding the number beside and below them. *(See Figure 14)* For instance, 4 is a number on the second diagonal and the square of it is 16. On the other hand, this can be computed by adding 6, the number besides it, and 10, the number below both number. This also works when one takes another example, such as 5. The square of 5 is 25 and it can be resulted by adding 10 and 15. This is a obscure but amazing pattern hiding in the Pascal's Triangle.

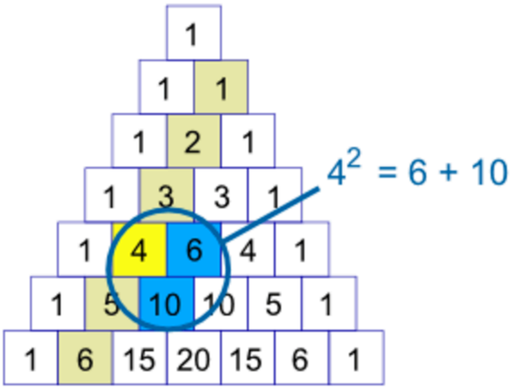

Figure 14: squares and addition

Lastly, the Fibonacci sequence can also be found in the Pascal's triangle. If we change the triangle a bit and make a right triangle and then cross the triangle over to add the numbers on the crossing lines, we will get the Fibonacci sequence. Just as shown in Figure 15. This, again, fascinated mathematicians with how widely used and appeared Fibonacci sequence and Pascal's Triangle can be. Actually, sequences all tend to be connected to one another in a sense; especially when it comes to real life applications.

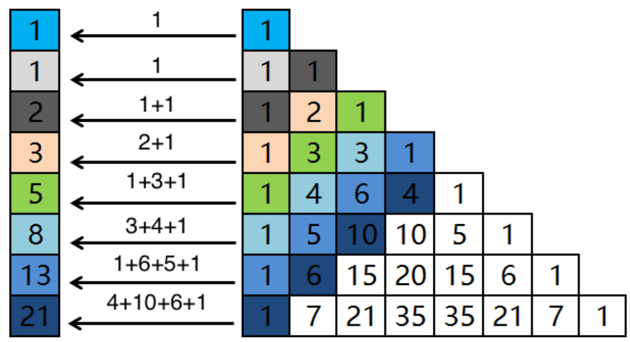

Figure 15: squares and addition

Pascal's triangle does not just seem to be fancy, but also has plenty applications. For example, Pascal's triangle can be used in expending polynomial functions. The nth power of the polynomial function corresponds with the nth line on the Pascal's triangle. Then we just have to fill in the Pascal's triangle numbers on that line in order as the coefficients of each power, from high to low. For instance, (x+1) to the power of three will have coefficients corresponding to the third line on the Pascal's triangle, which has numbers 1, 3, 3, and 1. Then from the highest power, which is x to the power of 3 to the lowest power which is x to the power of 0, each variable is match with 1, 3, 3, 1 in order. As a result, x to the power of 3 will have a coefficient of 1; x to the power of 2 will have a coefficient of 3; x to the power of 1 will have a coefficient of 3; and x to the power of 0 will have a coefficient of 1. This works with other powers as well. *(See Table 1)*

Power	Binomial Expansion	Pascal's Triangle
2	$(x+1)^2 = \mathbf{1}x^2 + \mathbf{2}x + \mathbf{1}$	1, 2, 1
3	$(x+1)^3 = \mathbf{1}x^3 + \mathbf{3}x^2 + \mathbf{3}x + \mathbf{1}$	1, 3, 3, 1
4	$(x+1)^4 = \mathbf{1}x^4 + \mathbf{4}x^3 + \mathbf{6}x^2 + \mathbf{4}x + \mathbf{1}$	1, 4, 6, 4, 1
	... etc ...	

Table 1: Polynomial expansion and Pascal's Triangle

Another application of the Pascal's triangle will be associating with permutation problem of head and tail. A typical permutation problem is all the possible outcomes of throwing a coin nth time. Exclude using the permutation calculations or tree diagram, it turns out that this can be found with the help of Pascal's triangle as well. The number of Head and Tail in the possible outcomes vary from pure head, to mostly head, to even (for even situations) then to mostly tail, and lastly all tail. In this order, the number of outcomes within each of these situations can be calculated by the Pascal's triangle. For example, if there are 3 tosses, the possible situations are "all head", "two head one tail", "two tail one head", "all tail". Since it is 3 tosses, we search for numbers on the third line of Pascal's triangle, which are 1, 3, 3, 1. Then, we pair these numbers to each of the situations in order. Finally, we will get 1

Tosses	Possible Results (Grouped)	Pascal's Triangle
1	H T	1, 1
2	HH HT TH TT	1, 2, 1
3	HHH HHT, HTH, THH HTT, THT, TTH TTT	1, 3, 3, 1
4	HHHH HHHT, HHTH, HTHH, THHH HHTT, HTHT, HTTH, THHT, THTH, TTHH HTTT, THTT, TTHT, TTTH TTTT	1, 4, 6, 4, 1
	... etc ...	

Table 2: "Head and Tail" and Pascal's triangle

outcome for the “all head” situation; 3 outcomes for “two head one tail”, 3 outcomes for “two tail one head”, and 1 outcome for “all tail” situation. This works only if the Pascal’s triangle numbers are paired to each set in order. *(See Table 2) [7]*

Conclusion

In conclusion, there 2 types of sequences and 4 specific interesting sequences being discussed in this paper. The two types being arithmetic sequence and geometric sequence; and the 4 specific interesting sequences being the Fibonacci sequence, the Catalan numbers; the polygon numbers and the Pascal’s triangle. These are just the tip of the iceberg of sequences. There are countless sequences existing and being created at every second. Sequence is much more than math. It is about living and thinking as humans. Utilizing sequences is a great improvement to our lives and a creative way of dealing with problems.

Footnote

[1]: MATHguide.com. "Geometric Sequences and Series by MATHguide." MATHguide Is the Source for Interactive Mathematics for Students, Parents and Teachers Called MATHguide. N.p., n.d. Web. 16 July 2017.

[2]: Boundless. "Applications of Geometric Series - Boundless Open Textbook." *Boundless*. Boundless, 28 Feb. 2017. Web. 16 July 2017.

[3]: Jason Marshall, PhD, The Math Dude May 5, 2010. "What Is the Golden Ratio and How Is It Related to the Fibonacci Sequence?" *Quick and Dirty Tips*. N.p., 09 Oct. 2013. Web. 16 July 2017.

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