Paper 2
The number e
Math of the Universe
Duke University
Kelly Tong
07/24/2017

The magical e

Introduction

Everyone who have undergone elementary school must have deep impression on the number Pi since it is closely related to circles and shapes. The number Pi, surely, is important to mathematics; however, there is another number, less familiar, which applies to more fields of mathematics and human lives. It is the number, e, known as the natural logarithm. e is so natural that it is everywhere in our lives. No matter it is the calculation for probability or for world population. No matter it is a combination problem or a compound interest rate financial problem. No matter it is the area rectangular hyperbola or the logarithmic numbers. There is always an e appearing somewhere. It is so natural that people do not realize it, but cannot stop experiencing every day.

In this paper, the number of e will be discussed based on its history, its occurrence in nature and its application in calculus along with the proving of Euler's identity. These specific fields of mathematical studies that are associated with e are interesting and important for developments of not only mathematics but civilizations as well.

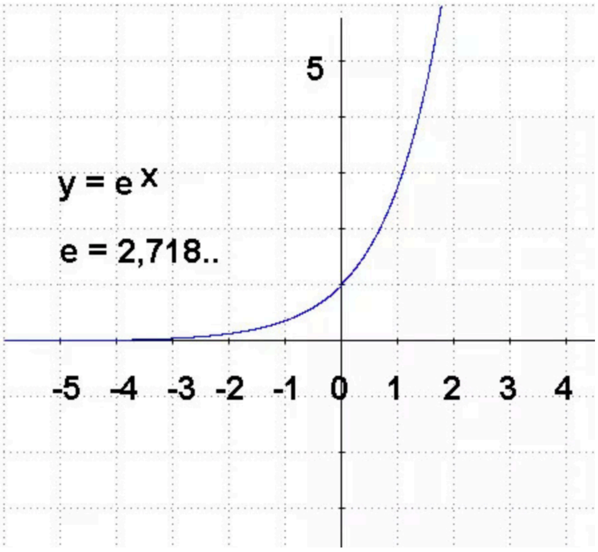
History of e

e was once discovered in 1618 when John Napier was doing a logarithmic table, although e was not directly included in the list of values, but the list of logarithms was actually calculated from this number. The second occurrence, possibly, was in 1647 when Saint-Vincent was dealing with computation of an area of a rectangular hyperbola. It is still debatable whether he realized the number e during the process. All the occurrences were, however, finalized by Euler as the natural logarithms, e. [1]

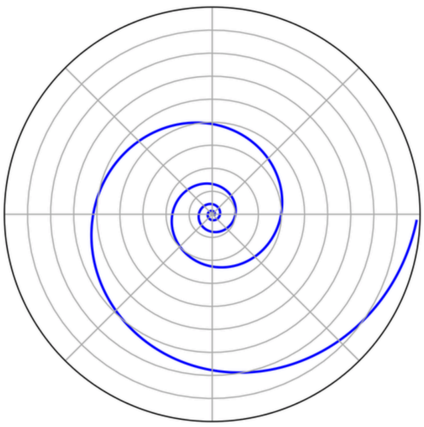
e is widely called as the natural logarithms due to its close relationships with the nature and humans' lives. Thousands of years ago, ancient Greek invented the word “nature” along with new perspectives of viewing the world. Since then, people started discussing about and solving issues occurring between humans and natures. e was discovered to be one of the most relatable number to human lives and nature. It appears almost everywhere. We can see it in hyperbola, in finances, in natural spiral, in calculus etc. It is so applicable and important for humans that the perspective of the world was changed by it. [2]

e in the nature

Since e is named the natural logarithm, it must have a very deep relationship with the nature. One of the most shocking facts is that e actually is related to spirals. Base e exponent function has graph that looks like the following:



This is the graph for the function e to the x. It looks pretty normal. However, exclude using this x-y axis system to represent the function, we can also use polar coordinates to represent the same thing. When we turn e to the x into polar coordinate, it becomes e to the theta, theta being the degree between the line and where it crosses at, and the graph looks like this:



In fact, this is called a logarithmic spiral. The degree between the line and the axis that it crosses at is always the same. Therefore, it is also called the equiangular spiral. It is already surprising that e forms a spiral that has equal angles, however, what amazed people more is that fact that this spiral formed by e to the theta is also related to the Fibonacci sequence. Ones who are familiar with Fibonacci sequence will know that it appears very often in natures, livings, especially spirals. Fibonacci sequence is a form of close approximation of the golden spiral, which forms in the golden rectangle of golden ratios. *(See Diagram 1)* The red curve is representing the golden curve and the green curve represents the close approximation forms by the Fibonacci sequence. The great Fibonacci sequence, obviously, is already very close to the golden curve (golden ratios) that everyone considers it to be the same. However, the actual golden curve is the same as a logarithmic spiral created by e, meaning that by comparison, e to the theta is closer to the golden curves than the Fibonacci sequence. This is absolutely surprising since Fibonacci numbers already occur in natural spirals of plants very frequently, and now people find out that e to the theta is even closer to the nature. [3]

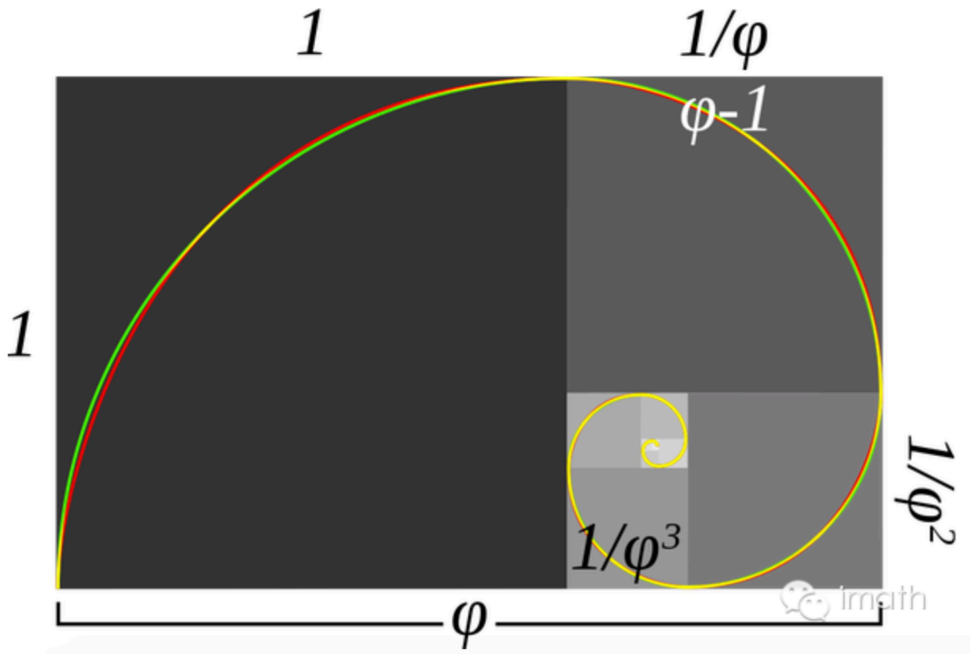


Diagram 1: Golden curve

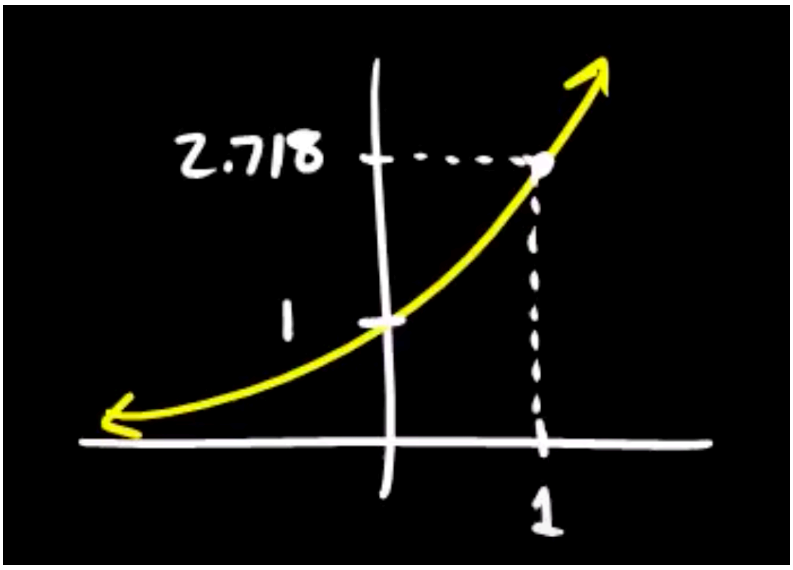
e in calculus

Property of base e exponent

Last but not least, e has a very unique identity and role in calculus with its nickname of "the one in calculus". This is because by differentiating a function of e to the power of x does not change anything, the derivative of the function is the function itself. Therefore, it is like the property of 1 which does not impact the result of any multiplication. In fact, function with e is the only type of function that has this property; this rareness makes it incredibly important. Its property can be concluded using the following expressions:

$$y = e^x \qquad\qquad \frac{dx}{dy} = e^x$$

Geometrically, it can also be expressed using the following graph:



By looking at both of these expressions, the equations and the graph, carefully, we can understand the property of e better. A lot of people know that the derivative of function e to the x is e to the x, but do not really understand the meaning of that. In calculus, by calculating the derivative of a function, we are actually calculating the a function of its slope. Hence, since the derivative of e to the x is the original function, it means that the y-value of the function equals its gradient. That is truly amazing. We can test it out by plugging in some of the number. [4]

$$\frac{dx}{dy} = e^0 = 1$$

For instance, on the graph, we can clearly see that the y-intercept of function e to the power of x is equal to 1; and by plugging in “x=0” into the derivative function, which gives us the slope, we also get 1. Another example would be plugging in “x=1” into the derivative function, we get the number e; and by looking at the y-value when x equals 1 on the graph, we also see that the value is e. This is how function of e to the power of x works. [4]

<u>Calculations of base e exponent</u>

Calculations associating with base e exponent are quite easy since e to the power of x never changes while being differentiated. However, there are 2 typical situations that need to be address. The first one is shown below:

$$f(x) = 3e^x \qquad f'(x) = 3e^x$$

This is a situation when the original function, which includes e to the power of x, does not change at all when being differentiated. This is because the number 3 positioning in front of e to the x is a coefficient; a coefficient never changes and can be put into the derivative directly following the law of differentiation. After that, e to the x, of course, will result in e to the x. Thus, the function and its derivative looks exactly the same. The second type of functions with base e exponent requires a little bit more work. It is presented as the following:

$$f(x) = e^{8x} \qquad f'(x) = e^{8x} \times 8 = 8e^{8x}$$

This situation includes a coefficient below the power of x. e to the power of a coefficient multiplies x can still be copied and pasted in the derivative, however, the coefficient, 8 for this case, has to multiplies the function again following the laws of differentiation. Therefore, the derivative will have an extra multiple of 8. [4]

Property of integral of e to the x

Differentiation and integration are two aspects of calculus. Differentiation is the process of decreasing the dimension of the function; on the other hand, integration is the process of increasing dimension. If there is a derivative, there must be an integral as well. The derivative of function e to the x is exactly the same as the original function; hence, the integral of e to the x is e to the x as well. These just undergo opposite process, and since e to the x is the derivative of itself, it must be the integral of itself. This property can be represented by the following expression:

$$f'(x) = e^x \quad \int e^x \, dx = e^x + c$$

The integral actually does not change anything, except there should be a "plus c" representing the integral constant. [4]

Calculation

The calculation with integral is more complicated, however, with function e, it is still quite straight forward. Take the following function as an example:

$$\int_2^4 e^x = e^4 - e^2$$

The "S" like symbol is the integral symbol and this asks for the value of function by having "x=4" minus "x=2". Since the integral of e to the power of x does not change, in this case, we just have to plug in 4 and 2 and minus one from the other. That is simple as long as the person distinguish which value should minus which one; it is one on the top minus the one at the bottom (usually the bigger one minus the smaller one). [4]

<u>Property of derivatives of logarithmic function</u>

This section will discuss logarithmic function with e base. This is not that easy anymore since the derivative does change this time. But before dealing with e base logarithmic function, it is important to understand logarithmic function and what they mean first. Logarithmic function often appears in the following generalization:

$$y = log_x^a$$

What this is asking is what power of x is equal to the value a, and the answer is y. This can be re-written into the following expression:

$$x^y = a$$

This is equal to the previous expression and is easier to understand. It just means that x to the power of y is equal to a. This is the foundation of logarithmic function calculations. So now we can investigate on e based logarithmic function which we name “Natural Logarithm”, abbreviated into “Ln”. [5]

In order to get the derivative of this e base logarithmic function, we need to simplify the logarithmic function as the first step. The function originally looks like this:

$$y = ln\,x = log_e^x$$

After simplifying it, it becomes like this:

$$e^y = x$$

This simplification is done by making both side of the equation exponent of base e. The left side becomes e to the power of y and on the right side e base logarithm cancels out with the extra e and leave x alone. e to the power of y equals x becomes the new function and is easier to calculate for the derivative. Firstly, we need to take the derivative of e to the power of y, the derivative of a function base e is itself so e to the power of y can be copied down; however, an additional derivative of the exponent y needs to be times to the equation as well. It becomes like the following:

$$e^y \times \frac{dy}{dx}$$

We can leave the left second for a second and get the derivative of the right side before moving to the next step. The right side is x, so the derivative, obviously, is 1. Then the equation becomes like this:

$$e^y \times \frac{dy}{dx} = 1$$

Now, we can divide e to the y to the right side and leave the derivative of y on the left side alone. Like this:

$$\frac{dy}{dx} = \frac{1}{e^y}$$

Previously we have known that e to the y is equal to x, so we substitute x into e to the y:

$$\frac{dy}{dx} = \frac{1}{x}$$

Our goal is to get the derivative of the natural logarithmic function, Ln(x), and we y is equal to that; therefore, the derivative of y is the derivative of the e base logarithmic function. The final solution is that the derivative of the e base logarithmic function is 1 over x presenting as following:

$$\frac{d}{dx} Ln(x) = \frac{1}{x}$$

Calculation

Based on the derivative of of the natural logarithmic function, we can do some calculations and understand it better. Take the following as an example:

$$y = ln\ (2x) \qquad \frac{dy}{dx} = \frac{1}{2x} \times 2 = \frac{1}{x}$$

Since x equals to 2x, the first thing we should do is just plugging in x=2x into the derivative function; then, we will get 1 over 2x. But there should be one more step, the coefficient before x needs to be included as well, we multiply the derivative function by 2. Finally, we will find out that the 2 will cancel out withe eh coefficient 2, then the answer will just be 1 over x. This tells us that the coefficient before x does not better, because whatever it is will just be eliminated through cancelling. This pattern can be utilized and whenever there is a coefficient before the x in base x logarithmic function, we can just conclude that the derivative of the function is 1 over x. [5]

Taylor Series

Taylor series is a special method that can be used to represent a function. It is defined as an infinite sum that gives the value of a function. Its general formula appears as the following:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0)+f^{'}(x_0)(x-x_0)+\frac{f^{''}(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\cdots$$

This formula expands an usual function into term an infinite sum which will eventually add up to be the original function. This can be very useful in specific applications since breaking a function down might be easier to calculate in those cases. By looking at the formula for Taylor series, we can see that it uses the original function, derivative, second derivative, third derivative……as the multiples which are multiplied to the fixed numbers for the formula. These fixed numbers are exponents of base x divide by factorials, such as x to the power of 0; x to the power of 1 divided by factorial of 1; x to the power of 2 divided by factorial of 2…… This method can be utilized to represent the function e to the x as well. Below is the expansion:

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!}
\end{aligned}
$$

By doing a comparison between the Taylor series general formula and Taylor series expansion for e to the x, one can see that they are very much the same and only derivatives were canceled. It turns out that since the derivatives of function e to the x stay the same forever, no matter it is the second derivative or the hundredth, when x=0 is substituted with the derivatives, the answer is always 1 (e to the power of 0 equals 1). Hence, the constant numbers in the Taylor series general formula all just multiply with 1 and stay in the equation without any changes being done. [6] [7]

There is also an interesting fact. The Taylor expansion of sine x and cosine x are exactly both half of the expansion of e to the x. Below are the expansion of sine x and cosine x:

$$
\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\end{aligned}
$$

$$
\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \\
&= \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}
$$

By looking at the cosine expansion equation, we can see that it takes the odd terms of e to the x expansion that include all even number squares. On the other hand, sine x expansion takes the even terms of e to the x expansion which include odd number squares. Thus, both of them are half of the e to the x expansion formula. This is quite interesting. [6] [7]

Euler's identity

In fact, there are more interesting and complicated relationships between sine, cosine and the number e. Euler's identity is one of the greatest mathematical formula and it includes relationship between these concepts along with imaginary number.

Euler's identity is named after Euler, a great Swiss mathematician who studied massive number of mathematical fields and made fascinating discoveries. This formula is one of the most well known and amazing formulas. Here is how it look like:

$$e^{\pi i} + 1 = 0$$

The discovery of this formula and the proving process require great apprehension with imaginary number, the number pie, e and polar coordinates etc. This section is going to undergo and explain the whole process with equations and diagrams.

Before starting, a few things about the reference plain need to be address. In our usual calculations of function, we use the x-y axis and coordinate. In this case, we name a point (x, y). However, a point on the plain can also be represented by a polar coordinate (a coordinate that uses the radius and degree to represent a point), (r, theta). *(See Diagram 2)* With this being said, if we see a

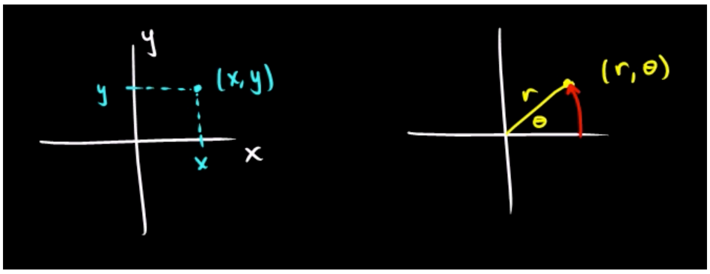
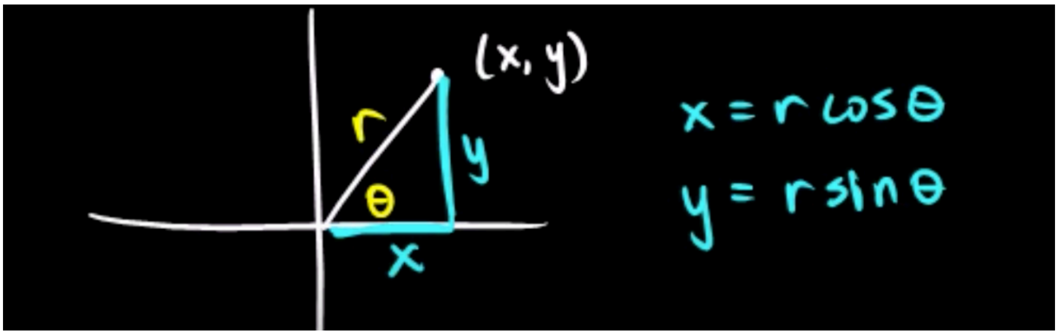


Diagram 2

point on the coordinate plain as a point on the circumference of a circle, then we can also name it as the following:



The x value can be written as radius multiplies by cosine theta and y value represented by radius multiplies by sine theta. [8]

The Euler's identity, similarly, was discovered based on this precondition, only that it is on the complex plain.

Normally, a point on the complex plain is named as (a, bi), just like (x, y).

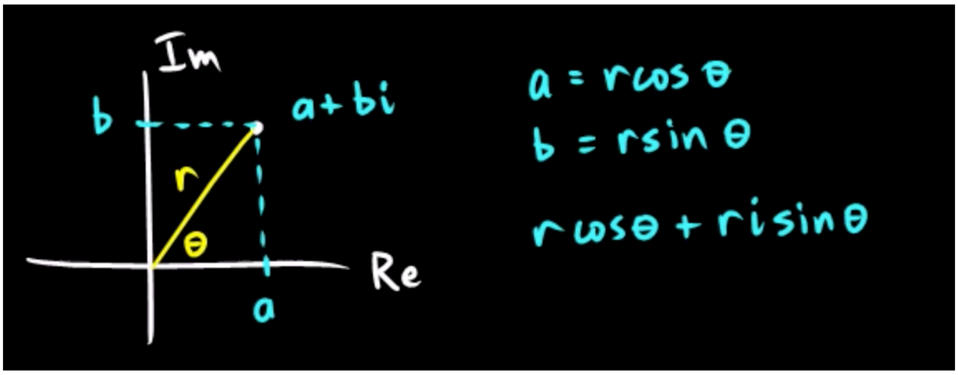
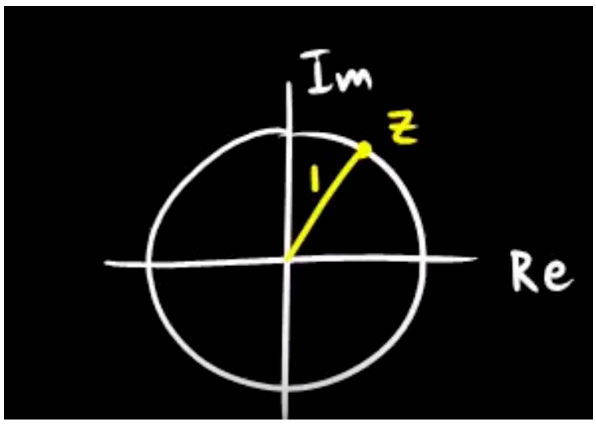


Diagram 3

However, like how (x, y) was transformed into polar coordinates, (a, bi) can also be written in polar forms. *(See Diagram 3)* A complex number can be named as (r cosine theta, ri sine theta). If we draw the plain again and make the point sitting on an unit circle, the coordinate will change again. Since the radius of an unit circle is equal to 1, the coordinate can be simplify to (cos



theta, i sine theta). In order to make later calculation easier, we will now name this point z. Naming this in polar coordinate is the very first step of finding the Euler's identity. [8]

After having the coordinate for z, we can write it in equation form as cosine theta plus i times sine theta equals z and take the derivative of that. The result and process is shown in Diagram 4. If we look at the result carefully, we will see that the one inside the bracket is exactly as the original equation of z.

$$z = \cos\theta + i\sin\theta$$

$$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$$

$$= i(i\sin\theta + \cos\theta)$$

Diagram 4

Therefore, we can substitute z in the equation and it becomes like this:

$$\frac{dz}{d\theta} = iz$$

Now we have z on both side of the equation; we need to position all the z on the same side. Then, we can take the integral of each side. After doing this, we will get Ln(z) on one side and "i times theta" on the other. Here is when e should be applied. We all know that Ln is a e base logarithm; therefore, by taking the e to the power of both side, we can get rid of the Ln and simply have z on the left side. The right side will then be e to the power of i theta. This is closer to the Euler's identity already and we have successfully used another representation for the point z. (See Diagram 5)

$$\frac{dz}{d\theta} = iz$$

$$\int \frac{dz}{z} = \int i\,d\theta$$

$$\ln z = i\theta$$

$$z = e^{i\theta}$$

Diagram 5

We know that z is equal to cosine theta, i sine theta, we can now substitute this with z and get the following equation:

$$\cos\theta + i\sin\theta = e^{i\theta}$$

This equation is actually a very important step towards the Euler's identity. After getting this equation, we need to change theta into Pi. Therefore, we can substitute Pi into all thetas. The equation, then, looks like this:

$$\cos\pi + i\sin\pi = e^{i\pi}$$

The right side is already part of the Euler's identity and we just need to compute for the left side which is calculable. Cosine Pi equals negative one and sine Pi equals zero which cancels the i out with it. We are now leave with negative 1 equals e to the power of i times Pi, which is known as the Euler's identity. It is more commonly written in "e to the power of Pi times i plus 1 equals 0." *(See Diagram 6)*

If $\theta = \pi$:

$$\cos\pi + i\sin\pi = e^{i\pi}$$

$$-1 + i(0) = e^{\pi i}$$

$$-1 + 0 = e^{\pi i}$$

$$\boxed{e^{\pi i} + 1 = 0}$$

Diagram 6

The Euler's identity is so fascinating because it includes five of the most important mathematical numbers, e, Pie, i, 1 and 0. It is such a mind blowing that these numbers are combined into a nice short equation like this. The proving process which includes polar coordinate and complex numbers also have shown Euler's unique and smart way of thinking. [8]

Conclusion

In conclusion, the number e has been discussed based on its naturalness and applications in calculus related problems. Specifically, the properties and calculations of e based exponent and base e logarithms were presented; and further investigation on Taylor series expansion was also included. Last but not least, an interesting proving process of the Euler's identity was explained. This paper focused on different aspects of the number e and has shown the significance of it.

Footnote

[1]: J J O'Connor. "The Number E." *The Number E*. N.p., Sept. 2001. Web. 24 July 2017.

[2]: Zhang, Ying Feng. "E: What Is Nature." *Zhi Hu*. N.p., n.d. Web. 25 July 2017.

[3]: Zhang, Ying Feng. "E: Amazing Spirals." *Zhi Hu*. N.p., n.d. Web. 25 July 2017.

[4]: Derekowens. "Calculus 6.2a - Calculus with Base E Exponents." *YouTube*. YouTube, 11 Feb. 2009. Web. 24 July 2017.

[5]: Derekowens. "Calculus 6.3a - Derivatives of Natural Logarithms." *YouTube*. YouTube, 11 Feb. 2009. Web. 24 July 2017.

[6]: Prof. Girardi. "Commonly Used Taylor Series." *Commonly Used Taylor Series* (n.d.): n. pag.*People.math.sc.* People.math.sc. Web. 25 July 2017.

[7]: Khanacademy. "Taylor Series at 0 (Maclaurin) for E to the X." *YouTube*. YouTube, 17 May 2011. Web. 24 July 2017.

[8]: Derekowens. "Calculus 6.11 - Euler's Identity." *YouTube*. YouTube, 02 Apr. 2011. Web. 24 July 2017.

Bibliography

Derekowens. "Calculus 6.11 - Euler's Identity." *YouTube*. YouTube, 02 Apr. 2011. Web. 24 July 2017.

Derekowens. "Calculus 6.2a - Calculus with Base E Exponents." *YouTube*. YouTube, 11 Feb. 2009. Web. 24 July 2017.

Derekowens. "Calculus 6.3a - Derivatives of Natural Logarithms." *YouTube*. YouTube, 11 Feb. 2009. Web. 24 July 2017.

J J O'Connor, and E F Robertson. "The Number E." *The Number E*. N.p., Sept. 2001. Web. 24 July 2017.

Khanacademy. "Taylor Series at 0 (Maclaurin) for E to the X." *YouTube*. YouTube, 17 May 2011. Web. 24 July 2017.

Prof. Girardi. "Commonly Used Taylor Series." *Commonly Used Taylor Series* (n.d.): n. pag. *People.math.sc.* People.math.sc. Web. 25 July 2017.

Zhang, Ying Feng. "E: Amazing Spirals." *Zhi Hu*. N.p., n.d. Web. 25 July 2017.

Zhang, Ying Feng. "E: What Is Nature." *Zhi Hu*. N.p., n.d. Web. 25 July 2017.