Schwarzschild Black Holes

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Mathematics of the Universe

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Abstracts:

This paper will focus on a variety of information about black holes, especially the Schwarzschild black holes, including but not limited to the intriguing background of discovery, the critical approach to solve them with equations, and the properties of them.

Early study:

A black hole is a spacetime region where the gravitational field is so strong that no information carrying objects and signals can escape it.[1] It has been fascinating the scientists for centuries. In the 18th century, physicist Laplace gave a solution of "black holes"—which did not get their name and concept as black holes at that time—based on Newtonian physics:

Suppose the mass of photon is m, the speed of light is c, and the mass and radius of a planet is M and r, respectively. So the photon ejecting from the planet has the kinetic energy(KE)

$$E_k = \frac{1}{2}mc^2,$$

[1] Introduction to Black Hole Physics, leri P. Frolov and Andrei Zelnikov 1.1.1

and the potential energy(PE)

$$E_p = GMm/r.$$

When the KE of photon is less than its PE, it cannot escape from the planet, so that this "planet" becomes a black hole.

$$E_k \leq E_p \quad \Rightarrow \quad \frac{1}{2}mc^2 \leq \frac{GMm}{r}$$

Solving this equation we can get

$$r \leq 2GM/c^2$$

Curiously, $r = \frac{2GM}{c^2}$ is exactly the correct solution to the radius of Schwarzschild black holes, viz. the Schwarzschild radius. While Newtonian physics does not apply to the spacetime around black holes, it, despite being a wrong method, enabled the earlier scientists to obtain the accurate answer. However, to uncover the mystery of black holes in a scientific way, we need general relativity and a nimble mind.

Background for Schwarzschild black holes:

In 1915, **Albert Einstein** developed his theory of general relativity, having earlier shown that gravity does influence light's motion.[2] This theory generalizes special relativity and Newton's law of universal gravitation, describing gravity as a geometric property of spacetime, which allowed scientists to predict and calculate cosmic phenomena far more accurately than before. According to general relativity, gravity is simply the curvature of spacetime.[3] Therefore, the black holes,

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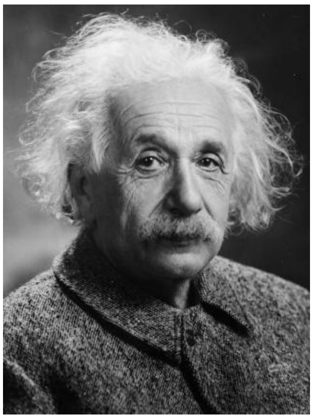
2 Schwarzschild, K. (1916). "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie". *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*. **7**: 189–196.

3 Introduction to Black Hole Physics, leri P. Frolov and Andrei Zelnikov 1.1.1

unlike other planets made of matter, were first regarded as “nothing” that were not observable and only lived in the equations until the discovery of Cygnus X-1 in 1971.[4]

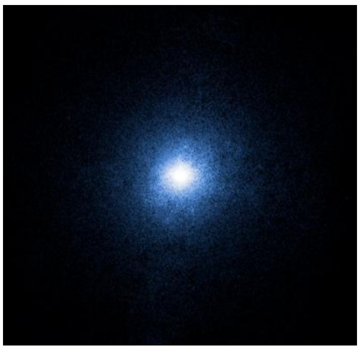


Albert Einstein [5]

Karl Schwarzschild [6]

[7]But the good news was: only a few months after general relativity’s publishing, the nimble mind emerged. **Karl Schwarzschild**, a German physicist and astronomer, worked out the first exact solution of Einstein’s filed equations in vacuum, on the assumption that the electric charge of the mass, angular momentum of the mass, and universal cosmological constant are all zero(non-rotating, non-charged). This solution was named after Karl Schwarzschild; so we have Schwarzschild black holes, **Schwarzschild radius** and **Schwarzschild metric**, which will be discussed in the following paper.

P.S. There’s a legend that Schwarzschild found the solution in the trench on the battlefield. But this remains a myth for trenches were never truly developed at the Russian Front.[8]



Cygnus X-1

[4] Kristian, J.; et al. (1971), "On the Optical Identification of Cygnus X-1", *The Astrophysical Journal*, **168**: L91–L93, Bibcode:1971ApJ...168L..91K, doi:10.1086/180790

[5] https://upload.wikimedia.org/wikipedia/commons/d/d3/Albert_Einstein_Head.jpg

[6] https://upload.wikimedia.org/wikipedia/commons/d/d3/Albert_Einstein_Head.jpg

[7]http://2.bp.blogspot.com/-FXXMtvLaMAY/TvzTzvqElVI/AAAAAAAAA5E/u44PYuHIS-k/s1600/381549main_cygX1_final_665.jpg

[8] Interesting aisde

Schwarzschild metric and radius[9]:

A Schwarzschild black hole is a **static** black hole that has no electric charge or angular momentum.[10] For such spherically symmetric and static body of radius R and mass M, we have a general metric as

$$ds^2 \quad = \quad \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu = U(r)dt^2 - V(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

where the metric components are

$$g_{00} = U, \quad g_{11} = -V, \qquad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta$$

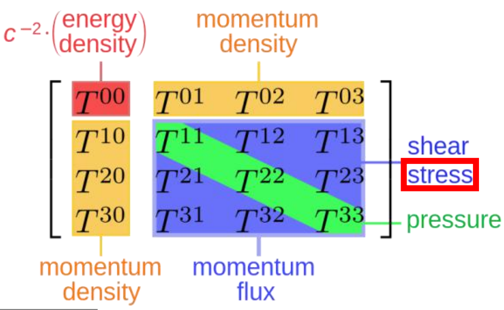
whose inverse is metric tensor with increased indices

$$g^{00} = \frac{1}{U}, \quad g^{11} = -\frac{1}{V}, \qquad g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2 \sin^2\theta}$$

Now that we have our metric, we can set out to find a suitable solution in Einstein's field equation. Here is the great equation(not in it's original form)

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

As has been mentioned above, the Schwarzschild black hole supposes **the cosmological constant** Λ to be zero. Moreover, $T_{\mu\nu}$, **the stress-energy tenor** vanishes in our metric (we only have g00,g11,g22,g33).



[11]

[9] Equations got from http://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf
[10] https://en.wikipedia.org/wiki/Schwarzschild_metric
[11]https://upload.wikimedia.org/wikipedia/commons/thumb/f/fe/StressEnergyTensor_contravariant.svg/805px-StressEnergyTensor_contravariant.svg.png

Then, the Einstein field equation can be simplified as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Feel much better with it? We are now halfway towards our goal.

Next, through **Christoffel symbols** and **Ricci tensor and scalar** (too difficult to clarify in this short paper), we can get R_{00}, R_{11}, R_{22}, R_{33}, four Ricci tensors, and R, the Ricci scalar. After substitute the unknown variables in Einstein's equation with all we have obtained, the miracle occurs:

$$V = \frac{1}{1 - \frac{C}{r}}$$

$$\boldsymbol{U = (1 - \frac{C}{r})}$$

Eventually, we substitute V and U with functions above in the general metric to get **Schwarzschild metric**:

$$ds^2 = (1 - \frac{C}{r})c^2 dt^2 - \frac{dr^2}{(1 - \frac{C}{r})} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

Well, it's not finished yet, because there is still a undetermined parameter C in the metric. Then some mass test is required to solve for

$$C = \frac{2GM}{c^2}$$

Congratulations! Finally, we acquire the full **Schwarzschild metric**:

$$\boldsymbol{ds^2 = (1 - \frac{2GM}{c^2 r})c^2 dt^2 - \frac{dr^2}{(1 - \frac{2GM}{c^2 r})} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2}$$

And the **Schwarzschild radius** is given as:

$$r_s = \frac{2GM}{c^2}$$

where G is the gravitational constant, c is the speed of light, and M is the mass of the body. If the body collapses to such an extent that its radius is less than r_s, the object will become a black hole.[12]

Also, note that the Schwarzschild metric above is only valid **outside** the Schwarzschild black holes ($R>r_s$); the interior solution must apply Interior Schwarzschild Metric, which needs to plus the time coordinate. To be strict, we should call the metric we have discussed in the paper the **Exterior Schwarzschild Metric**.

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Another form to show the Exterior Schwarzschild Metric

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

The best known form of Einstein field equations

12 Oxford dictionary of physics 6th ed.

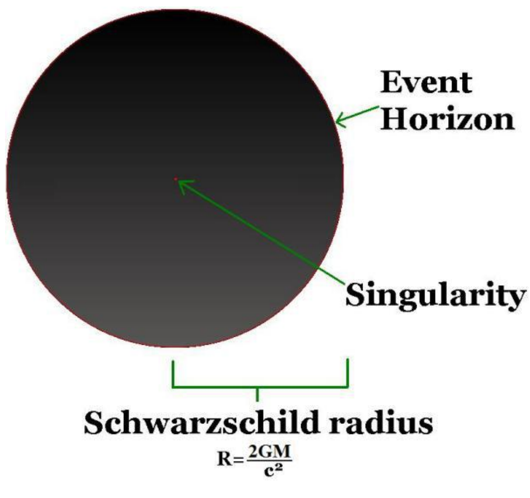
Some Schwarzschild black holes' structure and properties:

A black hole is a spacetime region where the gravitational field is so strong that no information carrying objects and signals can escape it.[13] A black hole is formed when the size of a gravitating object of mass *M* becomes smaller than

$$r_s = \frac{2GM}{c^2},$$

which is a pretty small number, since *G* is extremely small and *c* is relatively huge. And this boundary allowing no escape is called **event horizon,** the edge of a black hole. **Event horizon** gets its name because if an event occurs within the boundary, information from that event cannot reach an outside observer, making it impossible to determine if such an event occurred.[14]

Then, you may wonder what is in the innermost of a black hole. General relativity predicts that at the centre of the hole there is a **singularity**, a point at which the density becomes infinite and the presently understood laws of physics break down. This center, **singularity**, is also called **point singularity** in the non-rotating Schwarzschild black hole. The two pictures below may shed light on it.

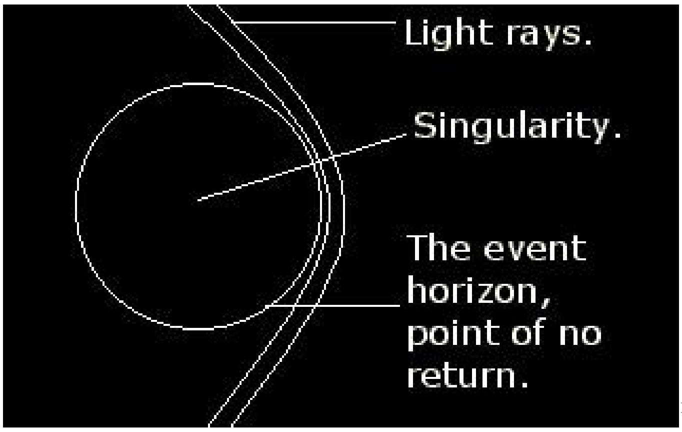


[15]

[13] Introduction to Black Hole Physics, leri P. Frolov and Andrei Zelnikov

[14] Wheeler, J. Craig (2007). *Cosmic Catastrophes* (2nd ed.). Cambridge University Press. ISBN 0-521-85714-7.

[15] https://upload.wikimedia.org/wikipedia/commons/b/b6/Black_hole_details.JPG

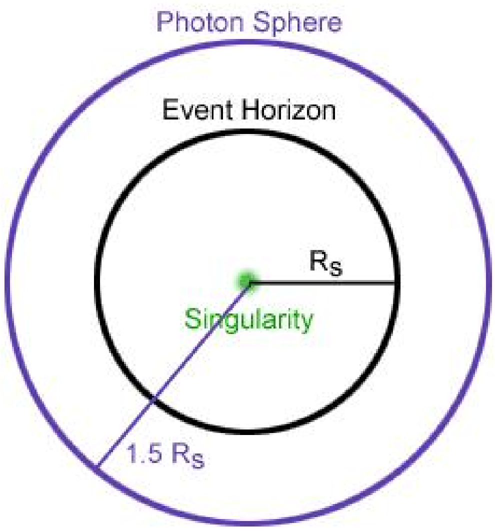


[16]

Also, a **photon sphere**, a spherical region of space where gravity is strong enough that photons are forced to travel in orbits, exists surrounding the black holes.[17] This sphere has a radius of

$$r = \frac{3GM}{c^2} = \frac{3r_s}{2}$$

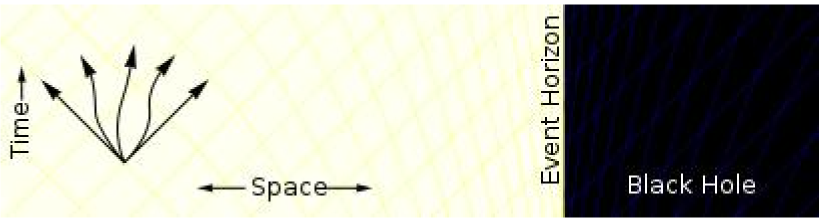
and it's like an aura showing the solemnity of black holes. Those imaginary pictures of black holes with bright silhouette really make some sense.



[18]

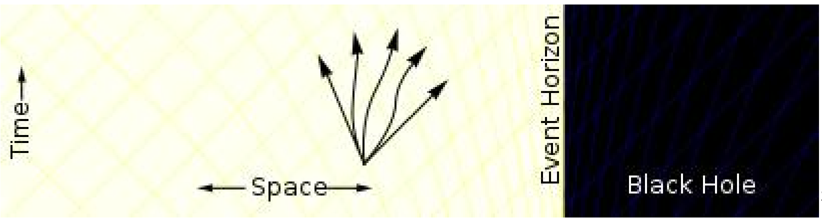
[16] https://qph.ec.quoracdn.net/main-qimg-2895f11e9f09f1449a9242fb1375b559
[17] Properties of ultracompact neutron stars
[18] http://www.gothosenterprises.com/black_holes/images/static.gif

Despite all its beauty and dignity, you'd never try to approach a black hole, if you could.



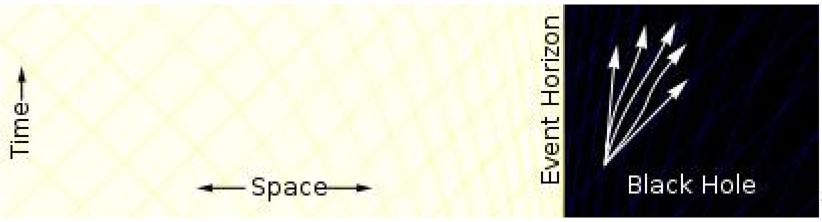
[19]

Far away from the black hole, you may move to any directions with a speed lower than light, just as your daily walk.



[20]

Closer to the black hole, the spacetime deforms gradually and attracts you with enormous force. But you still have the chance to escape at this time.



[21]

When you get inside the event horizon(suppose you survive the passing), you will be brought eventually towards the singularity. Under no circumstances can you escape.

However, we cannot endure such kind of deformed spacetime that the destiny of falling into a black hole is to be stretched into spaghetti, even way more thinner. The singularity is well protected from humans' vision.

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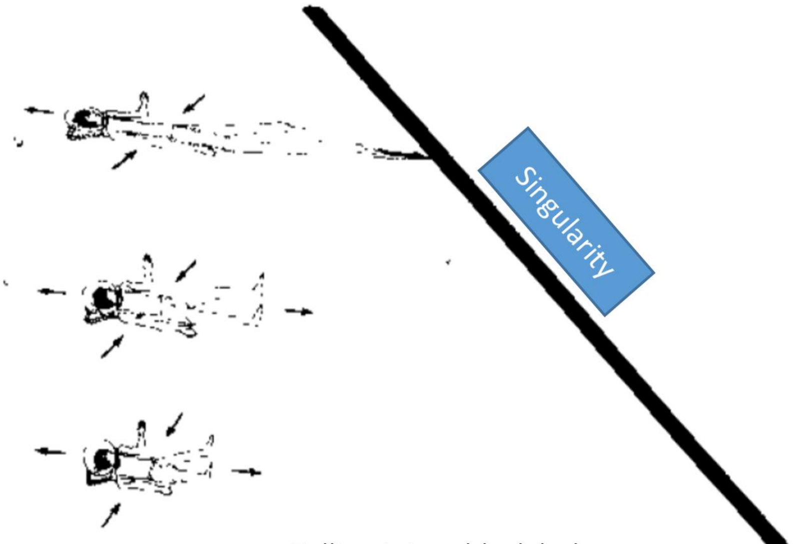
[19]https://upload.wikimedia.org/wikipedia/commons/thumb/5/55/BH-no-escape-1.svg/409px-BH-no-escape-1.svg.png

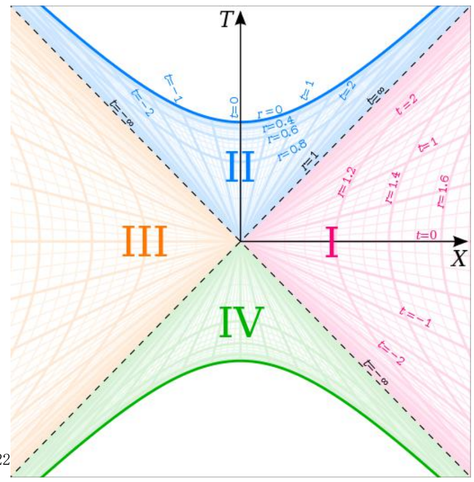
[20]https://upload.wikimedia.org/wikipedia/commons/thumb/1/10/BH-no-escape-2.svg/409px-BH-no-escape-2.svg.png

[21]https://upload.wikimedia.org/wikipedia/commons/thumb/8/87/BH-no-escape-3.svg/409px-BH-no-escape-3.svg.png



Falling into a black hole

But models about what is inside or through the singularity truly exist. For example, the **Kruskal Coordinates** cover the entire spacetime of the maximally extended Schwarzschild solution and sound plausible outside the physical singularity.



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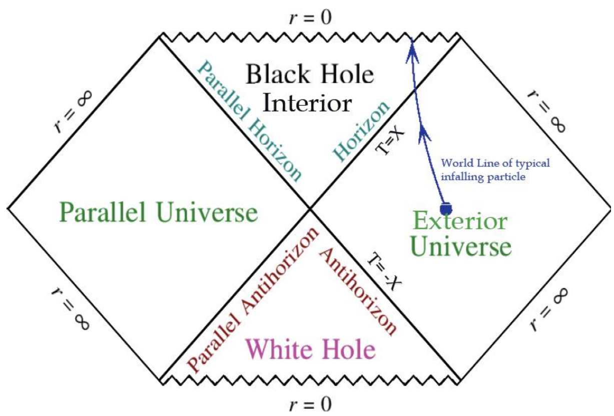

Kruskal Coordinates [23]

Through these coordinates, we can find some hypothetical regions such as **white hole** and **parallel universe**. But all these are based on the maximally extended Schwarzschild solution which may

[22] https://upload.wikimedia.org/wikipedia/commons/thumb/1/1c/Kruskal_diagram_of_Schwarzschild_chart.svg/512px-Kruskal_diagram_of_Schwarzschild_chart.svg.png

[23] https://inspirehep.net/record/866604/files/SchwKruskal.png

break down in another space just like the Newtonian physics breaks down in black holes. The conundrum of black holes still remains unsolved.

Conclusion:

Throughout the paper, we discuss the anecdotes, solutions and several properties of Schwarzschild black hole, which is actually is simplest type compared with all its brothers. In fact, such static black hole hardly exists in the universe; however, by knowing it, we can acquire fundamental knowledge about black holes as the basis for future learning. The exploration of black holes never stops.

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