Gravitational Wave

Kehan Chen

2017.7.29

Math 190S

Duke Summer College

1.Introduction

Since Albert Einstein released his masterpiece “theory of general relativity”, there has been prediction of the existence of gravitational wave. In Einstein's theory, gravity is treated as a phenomenon instead of a kind of force that Newton said, which results from the curvature of space-time.

As a gravitational wave passes an observer, we may find that spacetime around that observer distorted. Distances between objects increase and decrease rhythmically as the wave passes, at a frequency corresponding to that of the wave. [1] In that case, spacetime are stretched on one direction and compressed on the other.

On 11 February 2016, the LIGO (Laser Interferometer Gravitational-Wave Observatory) collaboration announced the detection of gravitational waves, on 14 September 2015, of two black holes. [2]

This discovery has made a stir to the whole field of physics and stimulated curiosity of many people including me. Thus, I am going to talk about how the gravitational waves are produced and detected in this paper.

2.What is gravitational wave?

2.1 Gravity

If we imagine that the spacetime is a sheet of rubber, things having mass will make the rubber sheet bend like a bowling ball on a trampoline.

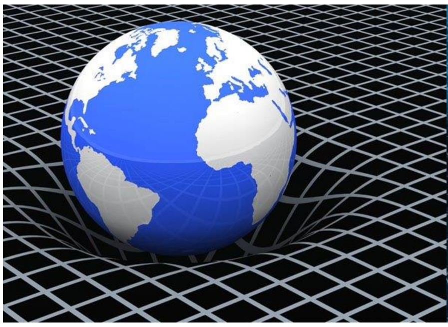

Figure 1

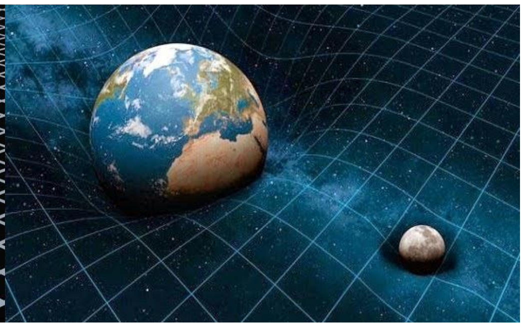

Figure 2

Generally, the more mass is contained within a given volume of space, the greater the spacetime will be distorted. Consequently, the motion in original spacetime may change because of the distortion. For example, the reason why Earth goes around the sun is that the sun is very massive, causing a big distortion of the spacetime around it. If you just try to move in a straight line around such a big distortion produced by sun, you will find yourself actually moving in a circle or eclipse orbit. That's how orbits work: there's not an actual force pulling the planets around, instead, a bending of the spacetime. [3]

So, it seems to be incredible that gravity is not a force. According to the theory of general relativity, it is the distortion of the space that traps our planet around the sun instead of the gravity force that we used to learn and think.

2.2 Where gravitational waves come from?

As objects with mass move around in spacetime, the distortion changes to reflect the changed locations of those objects. [4] In certain circumstances, accelerating objects generate gravitational waves in spacetime, which spread out at the speed of light. These phenomena are known as gravity waves. Unlike light or other kinds of waves that will be affected by the cosmic dust or gases while transmitting, gravitational waves will go through all spacetime smoothly while producing curvatures or distortion to spacetime.

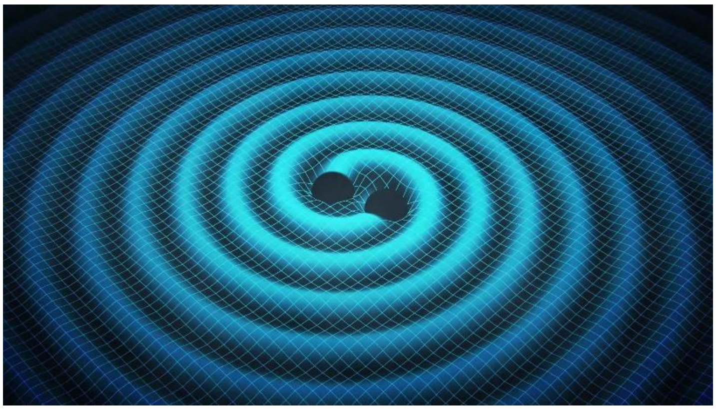

Figure 3

In fact, gravitational waves are the ripples in spacetime resulting from motion objects. In that case, everything with mass or energy can make gravitational waves. However, since gravity is very weak comparing to other kinds of force, only those having large masses and moving really fast can produce practically detectable waves such as a pair of rotating neutron stars or black holes.

Since we can’t generate detectable gravitational waves on Earth, the only way to study them is to look to the places in the Universe where they are generated by nature. The Universe is filled with incredibly massive objects that undergo rapid accelerations. [5] In general, the sources of radiating gravitational waves are binaries, black holes, supernova, rotating neutron stars and inflation.

2.3 Calculations about gravitational waves

Since detectable gravitational waves are produced by the accelerating objects having huge mass, there must be some relationship between the waveform of the gravitational waves and the property of the source.

Gravitational waves are now understood to be described by the theory of general relativity. In the simplest cases, the energy implications of gravitational waves can be deduced from other conservation laws such as conservation of energy or conservation of momentum.

The most basic form of source of gravitational waves is a binary system.

Suppose the mass of two rotating objects stars are M_1 and M_2, and the distance between them is L. Since there is always a spinning center of binary system and a constant angular velocity, we define the distance between the two rotators respectively as r_1 and r_2 and the angular velocity as ω. Thus, the linear velocity of two objects of the binary system will be v_1 and v_2. (Figure 4)

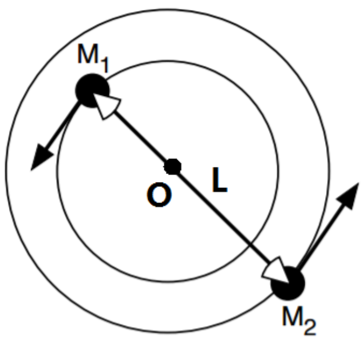

Figure 4

Using Newtonian mechanisms, we know that the gravity,F_G, between objects plays the role of centripetal force.

$$F_G = \frac{G \cdot M_1 \cdot M_2}{L^2} = M_1 \cdot \omega^2 \cdot r_1 = M_2 \cdot \omega^2 \cdot r_2$$

, where G is the gravitational constant.

Thus

$$\begin{cases} r_1 + r_2 = L \\ \frac{r_1}{M_1} = \frac{r_2}{M_2} \end{cases} \Rightarrow \begin{cases} r_1 = \frac{M_2}{M_1 + M_2} \cdot L \\ r_2 = \frac{M_1}{M_1 + M_2} \cdot L \end{cases}$$

So,

$$\omega = \sqrt{\frac{G \cdot M_2}{L^2 \cdot r_1}} = \sqrt{\frac{G \cdot (M_1 + M_2)}{L^3}}$$

For the object 1, its kinetic energy, E_{k_1}, is

$$E_{k_1} = \frac{1}{2} \cdot M_1 \cdot {v_1}^2 = \frac{1}{2} \cdot M_1 \cdot \omega^2 \cdot {r_1}^2 = \frac{G \cdot M_1 \cdot {M_2}^2}{2L \cdot (M_1 + M_2)}$$

And the situation is same for the object 2.

On the other hand, the potential energy of the binary system, E_p, is

$$E_p = -\mathrm{w} = -\int_L^{\infty} \vec{F} \cdot dL = -\int_L^{\infty} \frac{G \cdot M_1 \cdot M_2}{L^2} \cdot dL = -\frac{G \cdot M_1 \cdot M_2}{L}$$

, where w is the work needed to overcome the gravitational force, and move from its original space to infinite distance from it, where we define as a place that has zero potential energy.

Consequently, we can get the energy of the binary system, E_c (in conventional unit):

$$\begin{aligned} E_c = E_{k_1} + E_{k_2} + E_p &= \frac{G \cdot M_1 \cdot {M_2}^2}{2L \cdot (M_1 + M_2)} + \frac{G \cdot M_2 \cdot {M_1}^2}{2L \cdot (M_1 + M_2)} - \frac{G \cdot M_1 \cdot M_2}{L} \\ &= \frac{G \cdot M_1 \cdot M_2 \cdot (M_1 + M_2)}{2L \cdot (M_1 + M_2)} - \frac{G \cdot M_1 \cdot M_2}{L} \\ &= -\frac{G \cdot M_1 \cdot M_2}{2L} \end{aligned}$$

As these objects orbit, they generate gravity waves. Generally, gravitational waves carry energy away, and energy loss relativity predicts the rate at which the orbital energy is lost to this radiation. In conventional units, this rate, P_{E_c}, is [6]

$$\mathrm{P}_{E_c} = \frac{\mathrm{d}E_c}{\mathrm{d}t_c} = -\frac{32}{5} \cdot \frac{G^4}{c^5} \cdot \frac{(M_1 \cdot M_2)^2}{L^5} \cdot (M_1 + M_2)$$

, where c is the speed of light in vacuum and where the negative sign means that power is leaving the system, rather than entering.

To make the two equations consistent with our notation and to get rid of the constants G and c, we convert units of meters. Use the sloppy professional shortcut, "Let G = c = 1".

Thus, we get (in units of meters): [7]

$$E = -\frac{M_1 \cdot M_2}{2L}$$

$$\mathrm{P}_E = -\frac{32}{5} \cdot \frac{(M_1 \cdot M_2)^2}{L^5} \cdot (M_1 + M_2)$$

This equation involves a lengthy and difficult calculation starting from Einstein's field equations. So, I just quote this equation instead of deducing it from the start. With this equation, we can have a try to calculate the power emitted by the system of Earth and Sun. For this binary system, L is about 1.5×10^{11}m, and M_1 and M_2 are about 2×10^{30} and 6×10^{24} kg respectively. In that case, the power leaving the Earth-Sun system is about (negative sign only means that the power is leaving the space)

$$\begin{aligned}\mathrm{P}_E &= -\frac{32}{5} \cdot \frac{G^4}{c^5} \cdot \frac{(M_1 \cdot M_2)^2}{L^5} \cdot (M_1 + M_2) \\ &\approx -\frac{32}{5} \cdot \frac{G^4}{c^5} \cdot \frac{(2\times10^{30} \cdot 6\times10^{24})^2}{(1.5\times10^{11})^5} \cdot (2\times10^{30} + 6\times10^{24}) \\ &= -198.846\ (\mathrm{w})\end{aligned}$$

, which is a rather small quantity compared to the total electromagnetic radiation given off by the Sun (roughly 3.86×10^{26} watts).

Since there is energy loss in the binary system, there must be some influence to the state of motion of the binary system. Since the system loses energy like kinetic energy, the velocity of rotating objects will decline, causing the radius of their orbits to be shortened. The rate at which the radius changes is shown below. [8]

$$\mathrm{P}_r = \frac{dr}{dt} = -\frac{64}{5L^3}\cdot(M_1+M_2)\cdot M_1M_2$$

After talking about the binary system itself, I am going to talk about the gravitational waves produced by the system.

Gravitational waves are always described as ripples in the spacetime, so it will stretch the spacetime in one direction while squeezing the space time in the other direction. (Figure 5)

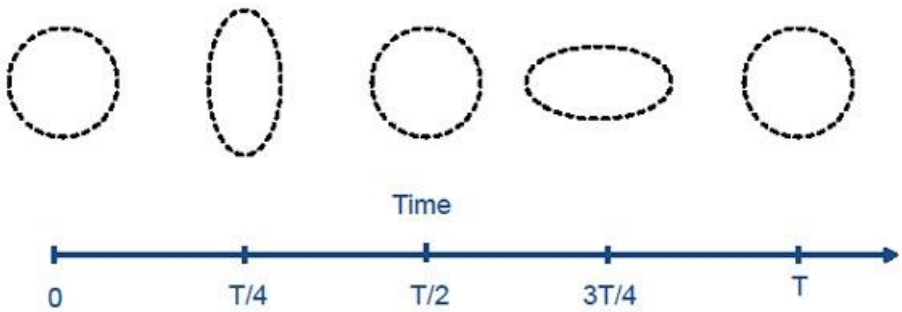

Figure 5

Just as electromagnetic waves, gravitational waves have two polarizations, commonly called h+ and h×. However, they are rotated by $45°$ with respect to one another as opposed to 90° because they correspond to a spin-2 field. The effect of the two polarization fields on a ring is illustrated in Figure 6. [9]

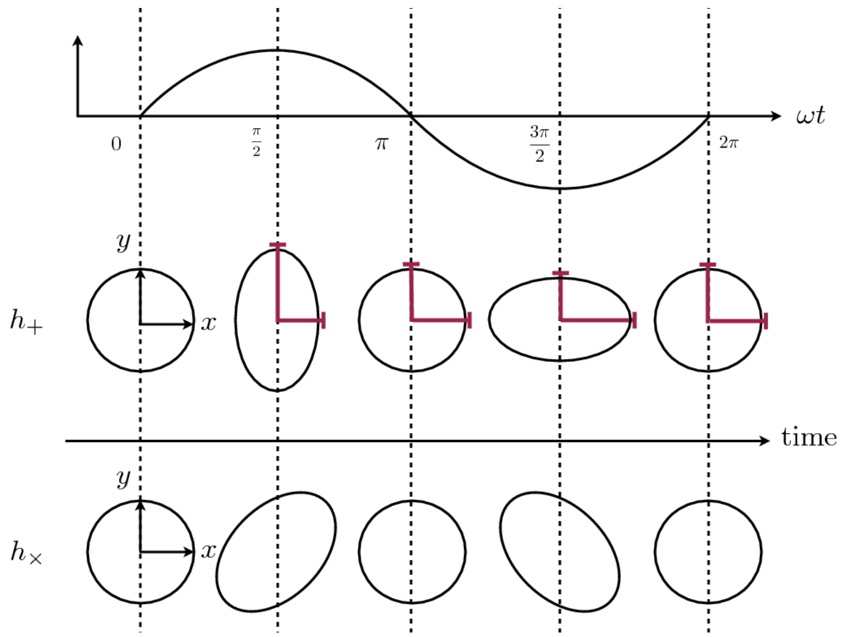

Figure 6

We can think in terms of the amplitude of the wave from a system in circular orbits. Let θ be the angle between the perpendicular to the plane of the orbit and the line of sight of the observer. Suppose that an observer is outside the system at a distance R from its center of mass. If R is much greater than a wavelength, the two polarizations of the wave related to the time, t, will be:

$$h_+ = -\frac{1}{R}\cdot\frac{G^2}{c^4}\cdot\frac{2M_1\cdot M_2}{L}\cdot(1+\cos^2\theta)\cos\left[2\omega\left(t-\frac{R}{c}\right)\right]$$
$$h_\times = -\frac{1}{R}\cdot\frac{G^2}{c^4}\cdot\frac{4M_1\cdot M_2}{L}\cdot(\cos\theta)\sin\left[2\omega\left(t-\frac{R}{c}\right)\right]$$

, where ω is the constant angular velocity of a circular orbit in Newtonian physics that I mentioned before in the paper: $\omega=\sqrt{\frac{G\cdot(M_1+M_2)}{L^3}}$. [10]

Now, take Earth-and-Sun system as an example again. Since we observers are in the plane of the orbit of the system, θ should be $\frac{\pi}{2}$ and $\cos\theta$ should equal to 0. Thus, $h_\times$ should be zero in the system of Sun and Earth system. And, we can figure out that

$$h_+ = -\frac{1}{R}\cdot\frac{G^2}{c^4}\cdot\frac{2M_1\cdot M_2}{L}\cdot(1+\cos^2\theta)\cos\left[2\omega\left(t-\frac{R}{c}\right)\right] = (-\frac{1}{R}\cdot 1.7\times10^{-10})m$$

Given that the minimum distance to find waves is R ≈ 1/4π light-year, the amplitudes will be h ≈ $10^{-25}m$. That is, a ring would stretch or squeeze by just one part in $10^{-25}m$. This is well under the detectability limit of all conceivable detectors. [11]

3.Detection of gravitational waves

On 11 February 2016, the LIGO collaboration announced the detection of gravitational waves from a signal detected at 09:50:45 GMT on 14 September 2015 of two black holes with masses of 29 and 36 solar masses merging about 1.3 billion light years away. [12]

We have calculated how weak the gravitational waves produced by the Earth-Sun system are. So how do the LIGO collaboration find those weak waves from the universe and a lot of interfering noises.

We know that the gravitational waves can stretch or compress the space that they go through. Nevertheless, if the space between two objects is distorted, we wouldn't notice it if we just use the common marks for measuring like marks on our ruler because these marks would also get distorted. But there is one ruler that never get stretched or compressed. However, there is one ruler that doesn't get stretched, one made using the speed of light. Given that light is constant in all reference system, if the space between two points gets stretched, then light will definitely take longer to go from one point to the other, and, similarly, if the space is squeezed, the time light takes to travel the distance will be shorter. This is where the LIGO experiment comes in.

When a gravitational wave comes through, it stretches space in one direction while squeezes space in the other direction as I mentioned before in the paper. To measure the stretching and squeezing, we return to a device called interferometer. [13] The equipment for measuring gravitational waves is like the one shown below.

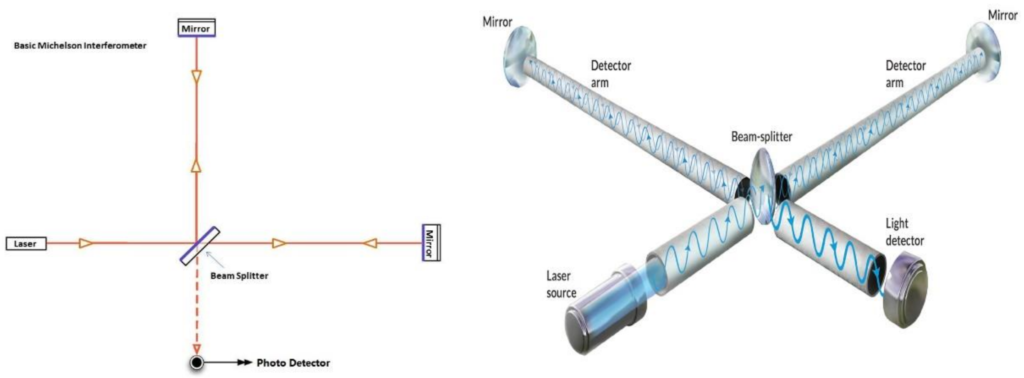

Figure 7

It has two precisely 4-kilometer-long tunnels which are perpendicular to each other. A laser beam is split and sent down a pair of long perpendicular tubes. The two beams bounce off mirrors and recombine at the Beam-splitter. Without the gravitational waves or distortion is space, the light waves come back lined up in such a way that they canceled each other out.

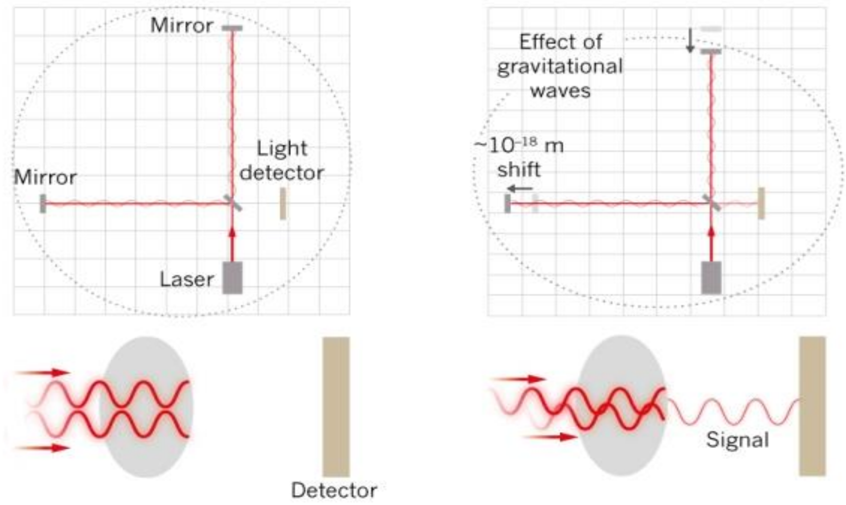

Figure 8

But when a gravity wave comes along, it distorts the space and changes the distance between mirrors and beam splitter. One arm becomes a little longer while the other becomes a little shorter. A instant later, they switch. This back and forth stretching and squeezing happens over and over again until the wave is passed. As the distances change, so does the alignment between the peaks and valleys of the two returning light waves and the light waves no longer cancel each other out when added together in the recombined beam. Now, some light waves do reach the detector with an intensity that varies as the distances of the tunnels varies. Measure that intensity, we are measuring gravity waves. [14]

By measuring the interference of lasers as they bounce between the different points, physicist can measure precisely whether the space is stretched or compressed.

Since the effect of a gravitational wave is so weak and easily confused with random noise, we need a smart data analysis technique. Scientists hope to identify the patterns of gravitational waves by comparing the wiggles they measure in the experiment with the waveform they expect from gravitational waves. [15]

However, even analyzed by smart devices, the wave may still not the gravitational waves because chances are that noise from Earth is likes the form of gravitational waves, and that's one of the reasons why two identical labs are built separately. The two LIGO sites, 3030 kilometers apart, in Hanford and Livingston will work in coincidence. [16]

Figure 9

If both labs pick up the same wave, chances are that it is not noise from Earth but a gravitational wave from space. The signal of the first observation of gravitational waves is shown below.

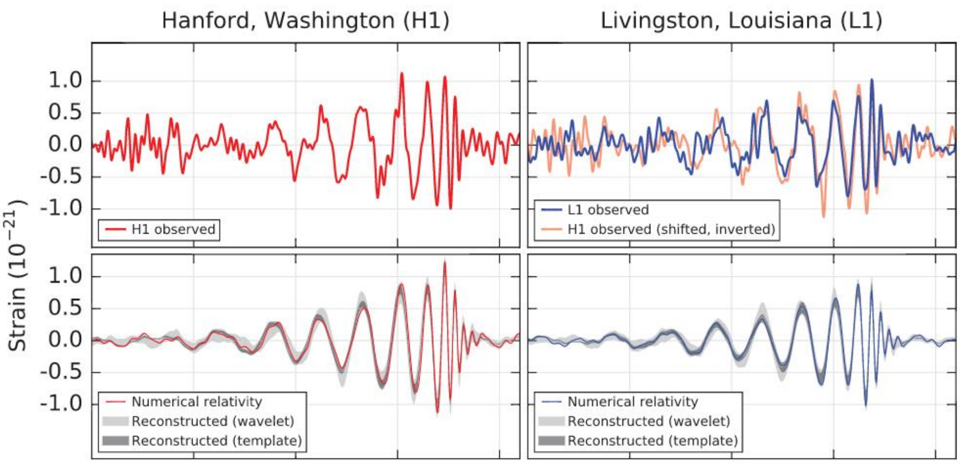

Figure 10

We can see that the two waveforms which are after analysis by devices are extremely similar after simulating into smooth lines. And we can say that they are gravitational waves from the universe instead of noise from our surroundings.

4.Conclusion

Before the detection of gravitational wave, human have always been blind for most of the universe. But now, we can actually hear the universe. We have talked about how gravitational waves are emitted according to general relativity and how they are detected. It is a completely new way of studying the universe. Hearing the gravitational wave can enable us to know more about the space and discover things that we did never expect. It is really about looking for new things that we didn’t know while examining the extreme of our knowledges and testing our theories about universe.

5.Bibliography

[1] [4] [10] [11] "Gravitational wave." Wikipedia. Wikimedia Foundation, 26 July 2017. Web. 29 July 2017.

[2] [12] Ghosh, Pallab. "Einstein's gravitational waves 'seen' from black holes." BBC News. BBC, 11 Feb. 2016. Web. 29 July 2017.

[3] [13] [15] PHD Comics. YouTube. YouTube, 01 Feb. 2016. Web. 29 July 2017.

[5] "Sources and Types of Gravitational Waves." LIGO Lab | Caltech. N.p., n.d. Web. 29 July 2017.

[6] [7] [8] Bertschinger Edmund, and Edwin F. Taylor. "Https://web.archive.org/web/20160129142844/http://www.eftaylor.com/exploringblackholes/GravWaves150909v1.pdf." N.p., 9 Sept. 2015. Web. 29 July 2017.

[9] Hinderer, Tanja Petra. Gravitational Waves from binary neutron stars and test particle inspirals into black holes. N.p.: n.p., 2008. Print.

[14] Bruno The Questionable. YouTube. YouTube, 30 Oct. 2006. Web. 29 July 2017.

[16] Barish, Barry C., and Rainer Weiss. "LIGO and the detection of gravitational waves." Physics Today 52 (1999): 44.