Paper 3
Nash Equilibrium
Math of the Universe
Duke University
Kelly Tong
07/31/2017

Nash Equilibrium

Introduction

Did you ever feel regret about doing something and feel stupid for your move? Have you ever wondered whether there is a best strategy for doing one thing? Are you eager to find out such strategy so you can eliminate large amount of mistakes? I am sure that no one can ignore or not be fascinated by the existence of such strategy. In fact, John Nash, one of the greatest mathematicians and economist, discovered and studied for this "best strategy". This strategy is known as the Nash Equilibrium.


Figure 1: John Nash

During his time at Princeton, John Nash, held great stress of producing a graduate paper. He had the perseverance for developing a totally original idea. He observed lives and actions, aiming for a pattern. He was discouraged by professors and mates about this belief. However, he did not give up. As being fictionalized in the movie, "A beautiful mind", he was enlightened by an encounter with several girls at the bar one night. John's friends at the bar wanted to approach to the most beautiful one among the girls and quoted Adam Smith quote, "with competition, the most worthy of them will rise to the top." At this point, John Nash denied his friends' and Adam Smith's statement by suggesting a better act and solution. He said that if every one of them approaches to the most beautiful girls, she will not accept anyone and her friends will not accept them as well because they do not want to be second; however, if they approach to the other girls and ignore the most beautiful one, then, everyone will have a pair and succeed. He denied Adam Smith's quote by saying that if everyone one chooses the best strategy for themselves and for the group, then there will be an equilibrium. It turned out that this idea was later developed into Nash equilibrium, an idea that is so central to economics and had won the 1994 Nobel Memorial Prize in Economic Sciences. [1]

In this paper, the definition and idea of Nash equilibrium will be discussed with illustrational examples and specifically about the prisoner dilemma. Next, ways to find and solve for a Nash equilibrium will also be presented. Lastly, dominant strategy which coexist with Nash equilibrium in economics will also be discussed, especially the connection between the two.

Definition

In order to understand the Nash equilibrium strategies, we need to know what is it first. According to the formal definition from Princeton University, "Nash equilibrium is a stable state of a system that involves several interacting participants in which no participant can gain by a change of strategy as long as all the other participants remain unchanged." This is a very long sentence but breaking it down will help us understand it easily. The first half of the sentence says that a Nash equilibrium exists in a system, like games, that involves more than 1 participant and has the same rules exerting on every participant. This is pretty simple. Then, the second half states that if there is a situation when the changing strategy of any participant does not gain as the other participants remain unchanged, there will be a Nash equilibrium. In other words, this Nash equilibrium is a state when no single act that is able to change the result of the group. It is an equilibrium because since a change does not gain anything, then the player will want to stay the same to at least keep the current status. This state of every player wanting to keep the same move is the very equilibrium point. [2]

With this definition, we can use an example to further explain the Nash equilibrium. Imagine there are two driver that want to pass the same road from two opposite directions. If both of the go at the same time, they will crash together; and if both of them stop, they will never be able to cross the road. Suppose there is no traffic law and both of the drivers know the given situation, is there any Nash equilibrium in this case? The answer can be found from analyzing the below payoff matrix. [3]

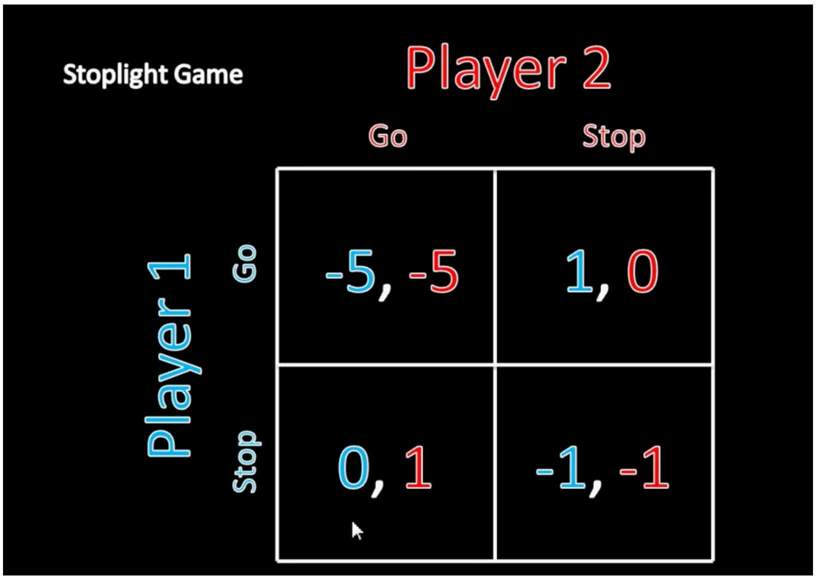


Diagram 1: Nash Equilibrium Example

As represented in Diagram, player 1 is driver 1 and player 2 is driver 2. The matrix concludes the state of both player when they go or stop. If both drivers go, they are going to end up with a crash, which is represented as -5 for both of them in the diagram. On the other hand, if both of them stop, they will lose time, which is represented as -1 for both of them in the diagram. These are not the equilibrium strategy because there are better choices for both drivers. If driver 1 goes first, then driver 2 has to wait; this is represented as 1 for driver 1 and 0 for driver 2. Conversely, if driver 2 goes first, driver 2 will have to wait and it results in 1 for driver 2 and 0 for driver 1. These two states are actually the equilibrium status, since each driver will only have to wait for a bit of time for the other driver to leave first and will not want to cause a crash or wait forever. We can check by assuming driver 1 is going first, then driver 2 will absolutely stop, because if he also go, he will end up with a -5. 0 is obviously better than a -5, therefore, he will choose to stop. Player 1, then, will not change the move either since he is already gaining 1 by going first. In this situation, both drivers will not want to change given what they will do. It also plays the same result when driver 2 leaves first and driver 1 waiting. Hence, there are two equilibrium strategy in this game, which is “Driver 1 goes, Driver 2 stops” and “Driver 2 goes, Driver 1 stops”. Through this example, one can understand that at a Nash equilibrium status, no participant will want to change his or her strategy given what the others are doing. [3]

Prisoner’s Dilemma

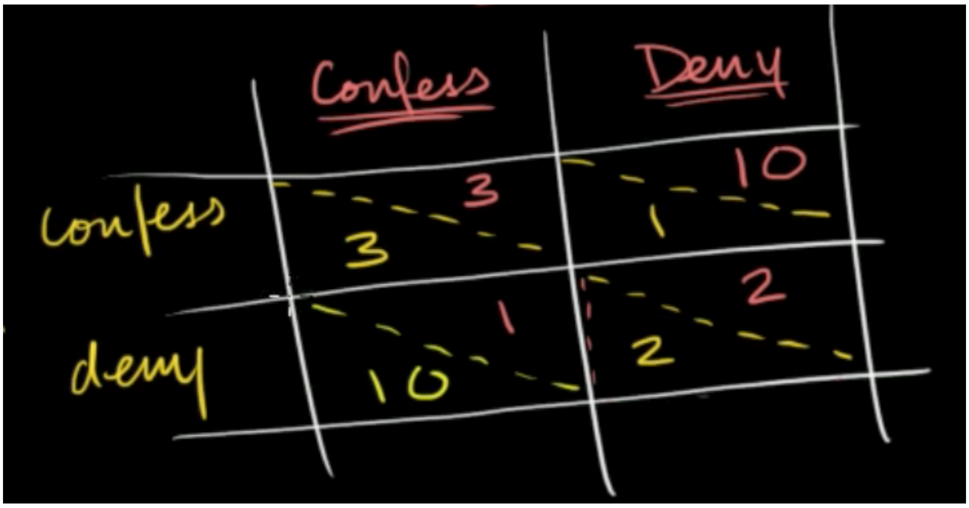
A famous game theory puzzle known as the Prisoner’s Dilemma is also related to Nash equilibrium. In fact, by using Nash equilibrium, one can find the best strategy and the stablest state for both prisoners.

Firstly, the Prisoner’s Dilemma describes a situation when:

One day, two criminals were captured for two separate cases and they were both sentenced for 2 years in the prison. However, the police suspected that they also did another crime together. There was no hard evidence, so the police try to persuade the prisoners to confess. He tells every prisoner that if only one of them confess, the one confess will get a penalty reduction of 1 year and the other prisoner will get 10 years as the penalty. If both of them confess, then both of them will get penalty lasting 3 years. If both of them deny, the penalty will remain 2 years for each one of them. Notice that this information was known by each of the prisoner. There is actually a Nash equilibrium status in the Prison’s dilemma. We can create a payoff matrix to help us find the equilibrium status. Like the following:

Prisoner 2

Prisoner 1



Let’s go through each status to see which is the equilibrium state. Firstly, if both player denies, then they will end up with 2 years of penalty. At this state, player 1 can change to confess and easily gets a 1 year penalty reduction if player 2 remains the same. Player 2 can also do the same to gain advantage for himself. Since the change of one player’s strategy will help him gain when the other player remains unchanged, this does not line up with the definition of Nash equilibrium; and thus this states is not an equilibrium state. Then we can look at the state when Prisoner 1 confesses and Prisoner 2 denies. In this case, Prisoner 1 only gets 1 year of penalty and Prisoner 2 gets 10 years. Of course, Prisoner 2 will be very unhappy and feel unfair; he will surely also confess to reduce his penalty year to 3 years. Since the change of strategy of Prisoner 2 will gain him 7 less years of penalty, this is not an equilibrium strategy. The situation when Player 2 confesses and Prisoner 1 denies is the same, except that now Player 1 will change his strategy to get less penalty. Lastly, we can analyze the situation when both prisoners confess. This is the situation when both prisoners will not want to change their strategy because they will not be able to gain anything. If Prisoner 1 changes the his strategy to denying, then he will get 10 years penalty while Prisoner 2 gets 1 year, and vice versa. In this case, no one will want to change the strategy, thus it is the Nash equilibrium of the Prisoner's dilemma. [2]

If you look at the result carefully and compare it with all the other results, you can see that it is not the best strategy. The best strategy for both prisoners is when both of them deny so no one will get any extra penalty. However, people do not believe each other easily, especially under this kind of circumstance. Although both of the prisoners denying is the best strategy in terms of outcome, it is not the most stable and reliable strategy. Additionally each prisoner also has a personal optimal strategy, which is the other player chooses to deny when he chooses to confess. This is the optimal strategy for an individual prisoner but not a stable equilibrium status. Since the other prisoner, certainly, will not sacrifice himself for the good of other, this is not a stable status. All in all, there is one equilibrium status in the Prison's Dilemma, being both prisoner confessing. [2]

Method of finding Nash Equilibrium

In previous sections, the paper has already used two examples to explain what is Nash equilibrium and present the process of finding it. In this section, clearer approach to finding Nash equilibrium will be listed in steps.

The first step of finding a Nash equilibrium is turning the given situation into a payoff matrix. *(See Diagram 3 for what is a payoff matrix)* The rows should be representing all possible moves of one player and the columns should be representing all possible moves of another player. For instance, in a "rock

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

Diagram 3: Payoff matrix of "rock paper scissor"

paper scissor" game, each player can either do rock, or paper, or scissor. These are the 3 possible moves. So the 3 possible moves of player 1 should be labeled to each row, the 3 possible moves of player 2 should be labeled to each column. *(See Diagram 3)*

After producing the payoff matrix with correct labels and scores written down, we can check to see what is the best outcome for row player (in this case player 1) to the different strategies of column player (in this case player 2). The below diagram shows how this works. [4]

	Buzz swerves	Buzz doesn't swerve
Jimbo swerves	(3, 3)	(2, 4)
Jimbo doesn't swerve	(4, 2)	(1, 1)

In this situation, there are two players, Jimbo and Buzz. They each have 2 possible moves, which is to swerve or not to swerve. Jimbo is the row player, so we will check the best response for him in turns of different moves of the column player, who is Buzz. When the column player swerves, Jimbo's best response is to not swerve since it gains 4 points instead of 3 for Jimbo. *(See what is circled)* When the column player does not swerve, Jimbo's best response is to swerve since it gains 2 points instead of 1. *(See what is circled)* So now we have figured out the two best outcomes for the row player, Jimbo, based on the column player's strategy. [4]

Moving on to the next step, we need to find out the best outcomes for the column player according to the varied strategies of the row player. Below is the example:

	Buzz swerves	Buzz doesn't swerve
Jimbo swerves	(3, 3)	(2, 4)
Jimbo doesn't swerve	(4, 2)	(1, 1)

In this case, when Jimbo swerves, Buzz's best outcome is gained by do not swerve. He gets 4 points instead of 3. *(See what is circled with green)* When Jimbo does not swerve, Buzz's better act is to swerve, which gives him 2 points instead of 1. *(See what is circled with green)* [4]

When the above steps are done, we can go back to the payoff matrix and find the Nash equilibrium easily. The state when both players get their better outcome is the Nash equilibrium since they will not be changing anymore. In this case, there are two Nash equilibriums. One being Buzz swerves and Jimbo does not swerve, the othe being Jimbo swerves and Buzz does not swerve. Simply, the state that includes two circles is a Nash equilibrium state if you follow the previous steps. [4]

<u>Simplify Steps:</u>

1. Create a payoff matrix of the situation
2. Find the best outcome for row player for different strategies of the column player and circle it
3. Find the best outcome for column player for different strategies of the row player and circle it
4. The state with two circles is a Nash equilibrium

[4]

Further Investigation

We have already discussed about Nash equilibrium and how to find it. However, there is time when a game seems to have no Nash equilibrium. Some will just ignore it and treats it as a game with out a best strategy. But! As John Nash had stated, "there must be at least 1 Nash equilibrium in any finite game." (This becomes the very foundation of economics.) Therefore, in finite games, if you cannot find a Nash equilibrium following the methods listing in previous section, then, probably you have encountered a special equilibrium. This section will discuss about some special Nash equilibrium strategy.

Dominant Strategy

In some situations, one participant in the game might have a dominant strategy, which is "the best choice for player regardless of what the other player chooses." (Barron Ap economics definition) Usually, if only one participant has a dominant strategy, it seems that the other player cannot do anything and there is no Nash equilibrium. However, there is actually Nash equilibrium; just that it is being restricted by the dominant strategy of one player. We can understand this by analyzing the below situation. [5]

There are two firms, X and Y, and their pricing strategies affect each other's profits. They want to find the strategy that given what the other firm does, they will still gain the highest possible profit. The payoff matrix is presented as the following:

		Firm Y	
		High	Low
Firm X	High	50, 45	40, 35
	Low	40, 10	15, 20

(profits of x is underlined)

We can see that Firm x has a dominant strategy in this case, since no matter what Firm Y does, pricing high will always be good for it. If Firm Y charges high, it should also charge high since 50 is better than 40; if Firm Y charges low, it should also charge high since 40 is better than 15. It seems that Firm Y is somehow passive in this case and that it cannot change the result no matter of what. It is true that Firm Y cannot alter the fact that Firm X will charge high; however, since it knows that Firm X will charge high, it can also charge high. In this way, it gains benefits as well. It can also earn 45 as its profits, which is the highest among all 4 outcomes. Therefore, since both firms charging high is the only best outcome for both of them, it is the only Nash equilibrium in this case. [5]

We can conclude that usually when one participant has a dominant strategy, there will still be a Nash equilibrium (if it is a finite game) but will be limited. The Nash equilibrium will usually be less when there is an existence of a dominant strategy. [5]

Mixture Strategy

Another special type of Nash equilibrium is named the mixture strategy Nash equilibrium, which, just like how it is named, is the Nash equilibrium in mixture strategy. Mixture strategy is actually the next level of games, the first level being pure strategies. If there is a mixture of 2 or more pure strategies, it is kwon as the mixture strategy. More specifically, "a mixture strategy is the probability distribution over two or more pure strategies." Through analysis of a game with penny, we will be able to understand how mixture strategy Nash equilibrium works. The game works like this:

There are two players in the game. Each person has a penny and can choose which side of the penny they want to present. If the two pennies match, head and head or tail and tail, then player 1 gets 1 point and player 2 loses 1 point. If the two pennies mismatch, head and tail or tail and head, then player 2 gets 1 point and player 1 loses 1 point. This is a fair game since the probability of getting 2 match pennies and 2 mismatch pennies is the same. *(See Diagram 4 for the payoff matrix of this game.)* [6]

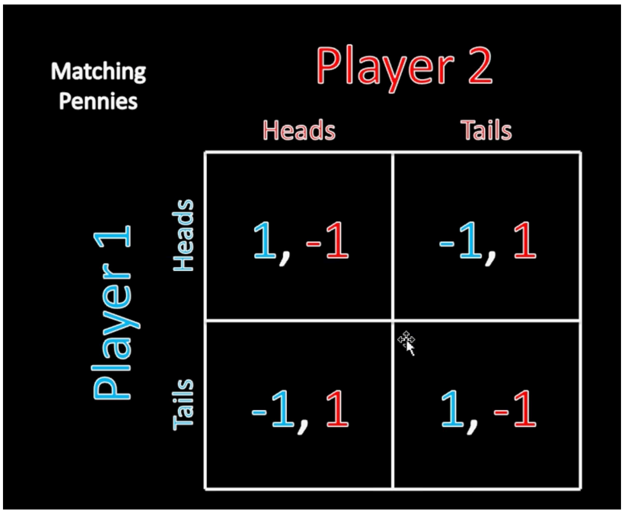


Diagram 4: Payoff matrix for "head tail"

If we try to find a Nash equilibrium state following the methods discussed, we will fail. Since the players can easily gain by the change of strategy given what the other player is doing, the previous steps do not work anymore. Now, we need to find out the Nash equilibrium for mixture strategy. Since a mixture strategy is the probability distribution over strategies, we need to find out the best probability of doing each act in order to get the Nash equilibrium. In this case, if the one player knows what the other is doing, the only way of not losing the game is to flip the coin and makes the decision random. The probability of getting a head is 1/2 and the probability of getting a tail is 1/2 as well. Therefore, if the player just flip the coin and makes the probability distributions even for head and tail, he will at least draw with the opponent. The final result will, then, be based on luck. In the end, this is the mixture strategy Nash equilibrium for this game. [6]

<u>Steps for finding a mixture strategy Nash equilibrium:</u>

1. Check if there is any Nash equilibrium for pure strategy. If not, do the following steps.
2. Compute for the best probability distribution for each act even the other player knows how to reed mind.
3. The final result will be the Nash equilibrium of mixture strategy. This should be in probabilities.

[6]

Sub-game Perfect Nash Equilibrium

Nash Equilibrium is a set of strategies that no participant will want to change the strategy given what the others are doing. This is quite important in economic sciences. However, sometimes, Nash equilibrium does not tell the whole story. Here is when sub-game perfect Nash equilibrium comes up. Sub-game perfect Nash equilibrium is a representation of Nash equilibrium of every sub-game that belongs to the same original game. A sub-game is a game that stores within an original game; it is sometimes quite unapparent. This section is going to discuss and explain sub-game perfect Nash equilibrium via the below example:

If a person named Y encountered a robber and is asked to give his money or he will be killed. The outcomes are organized into a payoff matrix presented below:

If the person, Y, gives the money to the robber, he is going to lose 10 points and the robber is going to gain 10 points. If Y is killed by the robber, he is going to lose 1 million points since death is much more severe than losing money. If Y refuses to give the money and the robber kills him, the robber will lose 10 thousand points since he gets a life sentence. [7]

Following the methods of finding Nash equilibrium, Y should choose to give the money if the robber is going to kill him; he should choose to not give the money if the

Diagram 5

robber chooses not to kill him. *(See circled answers in Diagram 5)* For the robber, he can either choose to kill or not kill when the person chooses to give the money; he should also choose

Diagram 6

to not kill the person if the person refuses to give the money. *(See circled answers in Diagram 6)* According to these steps, there are 2 Nash equilibriums in this situation. However, one of them does not really make sense. Although when the person, Y, gives the money to the robber, the robber gains 10 points, but he should also lose 10 thousand points if he chooses to kill the person. This information is not included in the payoff matrix and thus makes one of the equilibrium unstable. This is when sub-game Nash equilibrium

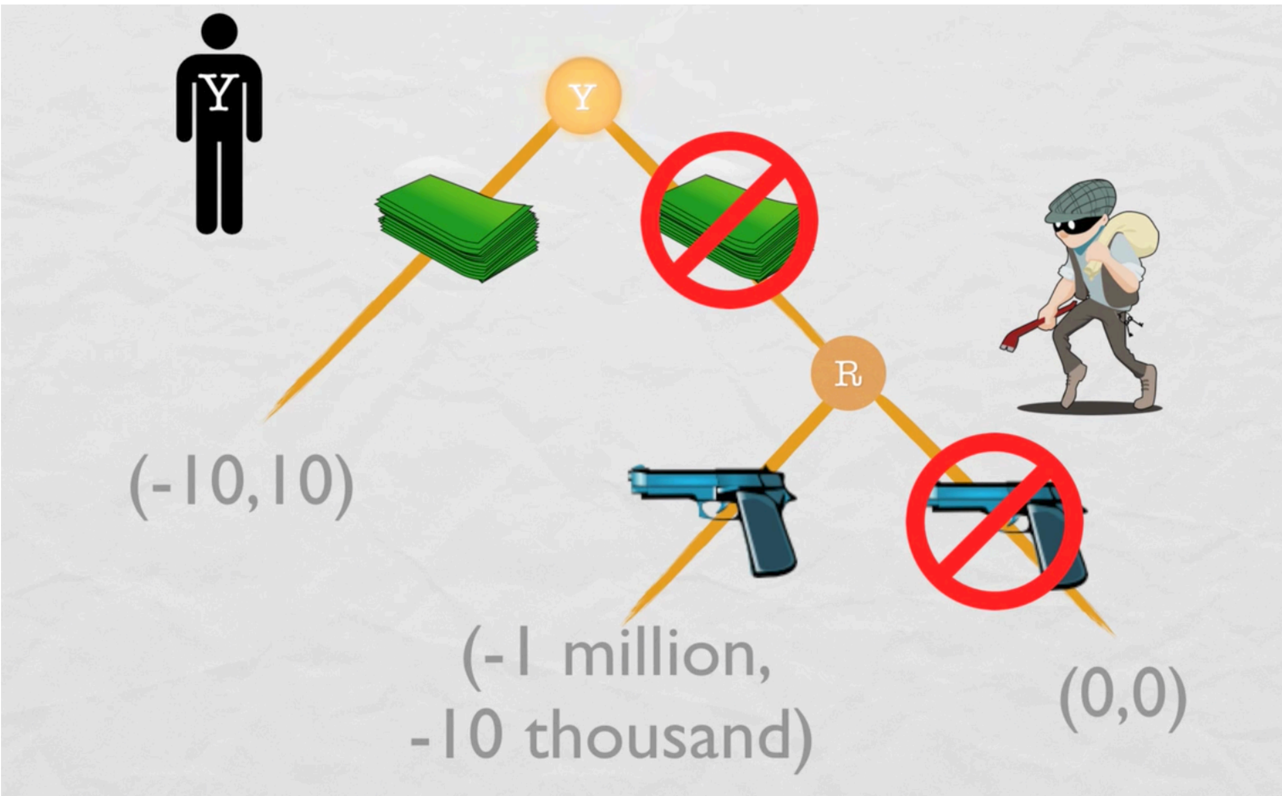


Diagram 7

comes in. In order to study sub-game Nash equilibrium, we need to change the payoff matrix into a tree diagram. *(See Diagram 7)* This diagram provides more information than the payoff matrix. We should look at the tree diagram from the top to the bottom. The game starts by the robber asking the person Y whether he will give his money to him or not. It ends when the person decides to give the money to the robber; in this case, the person loses 10 points and the robber gains 10. This makes more sense since if the robber has got the money already; it would be meaningless to kill the person. (Assume the robber only wants money) On the other hand, if the person refuses to give the money, then it moves to the sub-game, which the robber has a decision on killing the person or not. If so, we, then, should look at the outcome for the robber when he chooses to kill or not to kill, given the person refuses to give him the money. If the robber chooses to kill, he then will get a life sentence and lose 10 thousand points. If he chooses to not kill, he will just get 0. Obviously, 0 is much better than -10,000. Therefore, the robber should choose to not kill the person. Lastly, this state is known as the sub-game perfect Nash equilibrium, which means much more to this situation than the 2 Nash equilibriums for the whole game. [7]

Conclusion

In conclusion, Nash equilibrium is a very essential concept of economics and game theory. Its statement and calculation has helped enormous amount of people, especially economists, figuring out the best move under a situation. It might seem very basic, but it is like the bricks that build the tower, the height of the tower is depended on this fundamental material. Its idea has also shaped people's view of games differently. John Nash has spent his life aiming for more patterns and more original discoveries; and now, more and more people are studying his ideas and develop greater improvements for the society.

Footnote

[1]: "Nash Equilibrium in Economics: Definition & Examples Chapter 4 / Lesson 14 Transcript."*Study.com*. Study.com, n.d. Web. 31 July 2017.

[2]: "Game Theory and Nash Equilibrium | Microeconomics." *Khan Academy*. Khan Academy, n.d. Web. 31 July 2017.

[3]: JimBobJenkins. "Game Theory 101: What Is a Nash Equilibrium? (Stoplight Game)."*YouTube*. YouTube, 16 Jan. 2012. Web. 31 July 2017.

[4]: ProfRosenthal. "Finding a Nash Equilibrium: Example 2." *YouTube*. YouTube, 22 Aug. 2012. Web. 31 July 2017.

[5]: Frank Musgrave, Elia Kacapyr, and James Redelsheimer. "Imperfect Competition: Monopolistic Competition, Oligopoly." *AP Microeconomics/Macroeconomics*. 5th ed. N.p.: Barron's, n.d. 139-41. Print.

[6]: JimBobJenkins. "Game Theory 101 MOOC (#7): Mixed Strategy Nash Equilibrium and Matching Pennies." *YouTube*. YouTube, 02 Sept. 2012. Web. 31 July 2017.

[7]: GarrettPetersen. "Subgame Perfect Nash Equilibrium." *YouTube*. YouTube, 16 May 2013. Web. 31 July 2017.

Bibliography

Frank Musgrave, Elia Kacapyr, and James Redelsheimer. "Imperfect Competition: Monopolistic Competition, Oligopoly." *AP Microeconomics/Macroeconomics*. 5th ed. N.p.: Barron's, n.d. 139-41. Print.

"Game Theory and Nash Equilibrium | Microeconomics." *Khan Academy*. Khan Academy, n.d. Web. 31 July 2017.

GarrettPetersen. "Subgame Perfect Nash Equilibrium." *YouTube*. YouTube, 16 May 2013. Web. 31 July 2017.

[6]: JimBobJenkins. "Game Theory 101 MOOC (#7): Mixed Strategy Nash Equilibrium and Matching Pennies." *YouTube*. YouTube, 02 Sept. 2012. Web. 31 July 2017.

JimBobJenkins. "Game Theory 101: What Is a Nash Equilibrium? (Stoplight Game)."*YouTube*. YouTube, 16 Jan. 2012. Web. 31 July 2017.

"Nash Equilibrium in Economics: Definition & Examples Chapter 4 / Lesson 14 Transcript."*Study.com*. Study.com, n.d. Web. 31 July 2017.

ProfRosenthal. "Finding a Nash Equilibrium: Example 2." *YouTube*. YouTube, 22 Aug. 2012. Web. 31 July 2017.

<u>Image Sources :</u>
Gifford, Adam. "Payoff Matrix in Economics: Theory & Examples." *Study.com*. Study.com, n.d. Web. 31 July 2017.

Kenneth Singletary and Erica Goode Globe Staff and New York Times May 24, 2015. "John Nash, Winner of Nobel Prize in Economics, Dies in Crash - The Boston Globe."*BostonGlobe.com*. N.p., 24 May 2015. Web. 31 July 2017.