Euler’s Equation in Complex Analysis

$$e^{i\pi} + 1 = 0$$

Leqi Wang

July 2017

Math 190s Duke University

Euler’s Equation in Complex Analysis

Abstract

Euler’s equation is one of the most beautiful identities throughout the history. “ At first, you don't realize the implications, and then it hits you. It's the five most important constants, e, i, 1, 0 and π. All linked together. It's like the secrets of the cosmos distilled into a sonnet.” This paper states about the history concerning Euler’s identity in complex analysis, its proof (Taylor series),and how to understand it intuitively by means of group theory.

I. History

1. Leonhard Euler

figure1:1957 Soviet Union stamp commemorating the 250th birthday of Euler. The text says: 250 years from the birth of the great mathematician, academician Leonhard Euler.

Leonhard Euler was a Swiss mathematician, physicist, astronomer, logician and engineer back in the 18th century. Personally, he had 13 children with his wife Katharina Gsell. He also had phenomenal memory — in the 1730s, he lost his right eye and by 1771 he was completely blind, but it did not stop him from being productive. His mathematics works were know for high quality and quantity, his collected works, the Opera Omnia, contain 75 volumes and 25,000 pages and still coming out, the original editor's grandchildren was already old. And here is another astonishing indication of his work's quantity: there were paper still on his desk, and it took decades to clear the backlog, and after he died, Euler published 228 papers. And the quality of his work was also tremendous. "In mathematics and physics, there are a large number of topics named in honor of Swiss mathematician Leonhard Euler, who made many important discoveries and innovations. Many of these items named after Euler include their own unique function, equation, formula, identity, number (single or sequence), or other mathematical entity." In a website called " math world", if one input "Euler" in the search box, he or she would get 96 results — 96 mathematical terms that carry his name, 96 significant terms could be found in the dictionary（2011）. Just for comparison, here is the result if one input other mathematicians' name:

Euler: 96 entries

Gauss: 70 entries

Cauchy: 33 entries

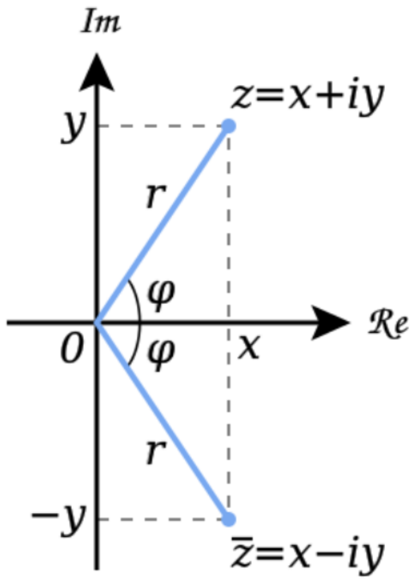
2. Euler's identity（1748）

$$e^{i\pi}+1=0$$

“Euler's formula is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometry function and the complex exponential function. Euler's formula states that, for any real number x, where e is the base of the natural logarithms, i is the imaginary unit, and cos and sin are the trigonometric function cosine and sine respectively, with the argument x given in radians. This complex exponential function is sometimes denoted cis x ("cosine plus i sine"). The formula is still valid if x is a complex number, and so some authors refer to the more general complex version as Euler's formula.” And therefore, this is the most incredible link among the 5 most important numbers in mathematics. “ If you are going to have a party, and you are going to invite 5 most important numbers to your party, who would you invite? We invite zero — the additive identity; we’d invite one, the multiplicative identity; if you want to do calculus you invite e; if you want to do geometry you invite π, if you want to do complex numbers you invite i.” And a dream team of these five figures are in the end linked by this fabulous formula in rather simple coincidence of Euler’s identity.

“Euler's formula is ubiquitous in mathematics, physics, and engineering. In 1988, a Mathematical Intelligencer poll voted Euler’s identity as the most beautiful feat of all of mathematics, and the physicist Richard Feynman called the equation "our jewel" and "the most remarkable formula in mathematics".

II. Complex numbers fulfill the axises in mathematics



The invention of numbers, in the history, was first used to count things. Of course, these numbers are positive integers, and it's not enough. Then it emerged 0 and negative numbers, which were laxly accepted by people. After that is the concept of rational number — the ratio among integers and integers. And then The square root of two was discovered, notably, it also triggered first mathematical crisis in the history. Up to now, there are all real numbers represented on the real number axis, and real numbers are successive. The axis of mathematics seems to be fulfilled — until the emergence of complex numbers.

"The solution in radicals of a general cubic equation contains the square roots of negative numbers when all three roots are real numbers, a situation that cannot be rectified by factoring aided by the rational root test if the cubic is irreducible. This conundrum led Italian mathematician Gerolamo Cardano to conceive of complex numbers in around 1545, though his understanding was rudimentary.

Work on the problem of general polynomials ultimately led to the fundamental theorem of algebra, which shows that with complex numbers, a solution exists to every polynomial equation of degree one or higher. Complex numbers thus form an algebraically closed field, where any polynomial equation has a root.

Many mathematicians contributed to the full development of complex numbers. The rules for addition, subtraction, multiplication, and division of complex numbers were developed by the Italian mathematician Rafael Bombelli. A more abstract formalism for the complex numbers was further developed by the Irish mathematician William Rowan Hamilton, who extended this abstraction to the theory of quaternions."

III. Taylor series

Since understanding Taylor series and Maclaurin series is essential for understanding the proof of Euler's equation, we'll briefly discuss this method named after the English mathematician Brook Taylor (1685–1731) and Scottish mathematician Colin Maclaurin (1698–1746).

If we start to We start by supposing that *f* is any function that can be represented by a power series:

$$f(x) = C0 + C(x - a) + C2(x - a)^2 + C3(x - a)^3 + \dots \dots \qquad |x - a| < R$$

The coefficients *Cn* must be in terms of *f*. To begin, notice that if we put *x a* in the equation above, then all terms after the first one are 0 and we get

$$f(a) = c0$$

Then we can differentiate the series in Equation 1 term by term:

$$f(x) = C1 + 2C2\,(x - a) + 3C3(x - a)^2 + 4C4(x - a)^3 + \dots \dots \qquad |x - a| < R$$

and substitution of x a in Equation 2 gives:

$$f'(a) = C1$$

Now we differentiate both sides of Equation 2 and obtain :

$$f''(x) = 2C2 + 2*3C3(x - a) + 3*4C4(x - a)^2 \ldots \ldots \qquad |x - a| < R$$

By now we can see the pattern. If we continue to differentiate and substitute x a, we obtain

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$. Thus we have proved the following theorem:

If f has a power series representation (expansion) at a, then its coefficients are given by the formula:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

So the Taylor series could be represented by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

For the special case *a* 0 the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

This case arises frequently enough that it is given the special name Maclaurin series.

IV. Proof of Euler's Equation

1. Using Maclaurin series

We've already known that Maclaurin series is one special case of Taylor series, and there is three Maclaurin series we could use to prove Euler's formula: sine, cosine, and e^z.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

If we substitute z=ix as the exponent of e, then we get:

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos x + i\sin x \end{aligned}$$

And when we substitute x = π, we have:

e^πi = cos(π)+i*sin(π)
= -1
or
e^πi+1 = 0
(Euler’s identity)

2. Using Calculus

We can set：

$$f(x) = cos(x) * i*sin(x),$$

then

$$\frac{df}{dx} = -\sin(x) + i \cdot \cos(x)$$
$$= i \cdot f(x)$$

And

$$\int (\frac{1}{f(x)}) df = \int i \cdot dx$$

$$\ln(f(x)) = ix + C$$

$$f(x) = e^{ix+C} = \cos(x) + i \cdot \sin(x)$$

And since f(0) = 1, C = 0:

$$e^{ix} = \cos(x) + i \cdot \sin(x)$$

V. Understand Euler's equation intuitively

There are many ways to understand formula $e^{ix} = \cos(x) + i \cdot \sin(x)$, which turns out to be one of the most confusing formulas in the world. It seems that each term in the formula make sense but the statement as a whole seems nonsensical. Maybe one is lucky enough to see what this means and some long formulas explaining why it works in a calculus class, but it still feels like dark magic. So let's change our basic point of view on the concept of numbers, and look them through the lens of Group theory, so that it'll become easier for us to understand why it’s true and why it makes intuitive sense. But notice, group theory is only a method help to understand Euler’s equation more intuitively but not a formal proof.

1.Group theory

“Group theory has three main historical sources: number theory, the theory of algebraic equations, and geometry... …In geometry, groups first became important in projective equations and, later, non-Euclidean geometry. Felix Klein's Erlangen program proclaimed group theory to be the organizing principle of geometry.” “The number-theoretic strand was begun by Euler, and developed by Gauss’s work on modular arithmetic and additive and multiplicative groups related to quadratic fields. Early results about permutation groups were obtained by Lagrange, Ruffini, and Abel in their quest for general solutions of polynomial equations of high degree. Évariste Galois coined the term "group" and established a connection, now known as Galois theory, between the nascent theory of groups and field theory.”

Essentially, “a group is collection of underlying relations, all associations between pairs of actions and the single action that is equivalent to applying one after other, is what makes a group.” One group has following properties:

Closure: $a, b \in G$ and $a*$(operation notation)$\in G$

Associative: $(a*b)*c = a*(b*c)$

Identity: $\exists e, a*e = e*a = a$

$$\text{Inverse: } \forall a, \exists a^{-1}, a*a^{-1} = e$$

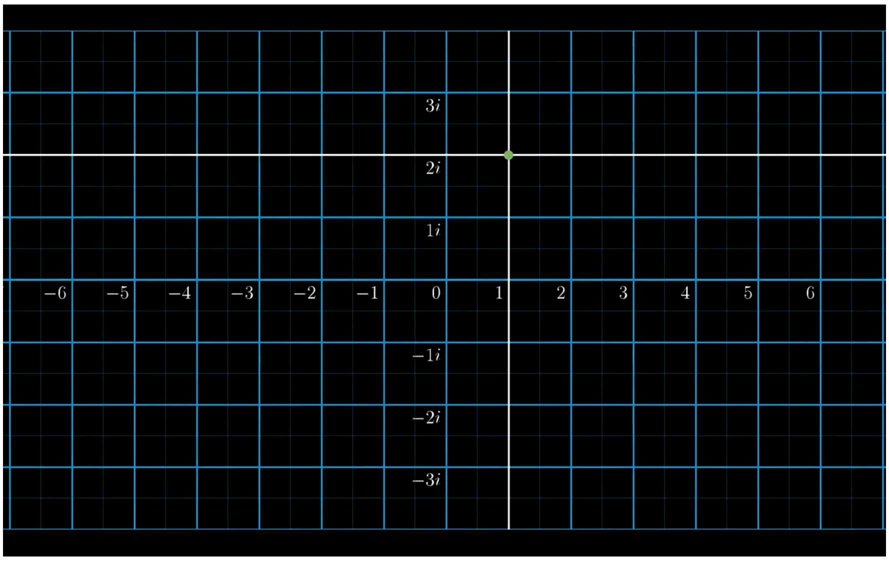
And if we want to understand Euler's equation, in the numerical case, we could consider the adding and multiplying operation as a sort of "action". In order to implement this action, we can reflect real numbers onto an axis, and imagine the addition(additive group of real numbers) and multiplication (multiplicative group of real numbers) as the action on the axis:

Addition = sliding to the right (Subtraction = sliding to the left)

Multiplication = stretching (Division = drawing back)

The arithmetic of adding umbers and multiply numbers is just one example of the arithmetic that any group of symmetries[1] has within it.

Now we can extend this idea, see what would happen when sliding&stretching actions occur on a complex plan:

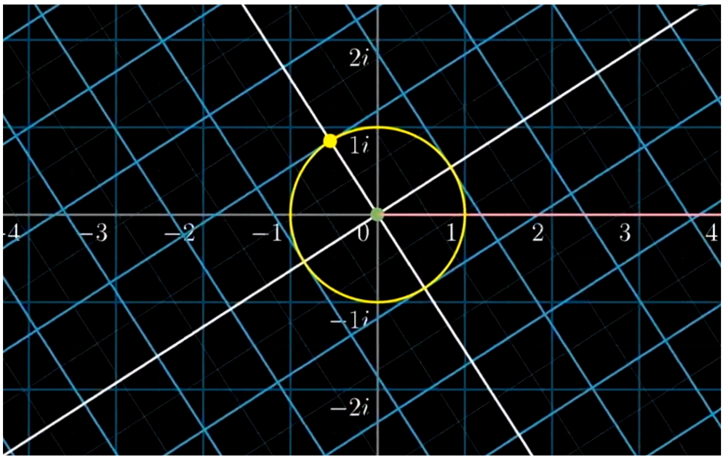


[1] https://en.wikipedia.org/wiki/Symmetry_group

The idea of addition can be thought of in term of successively applying actions.

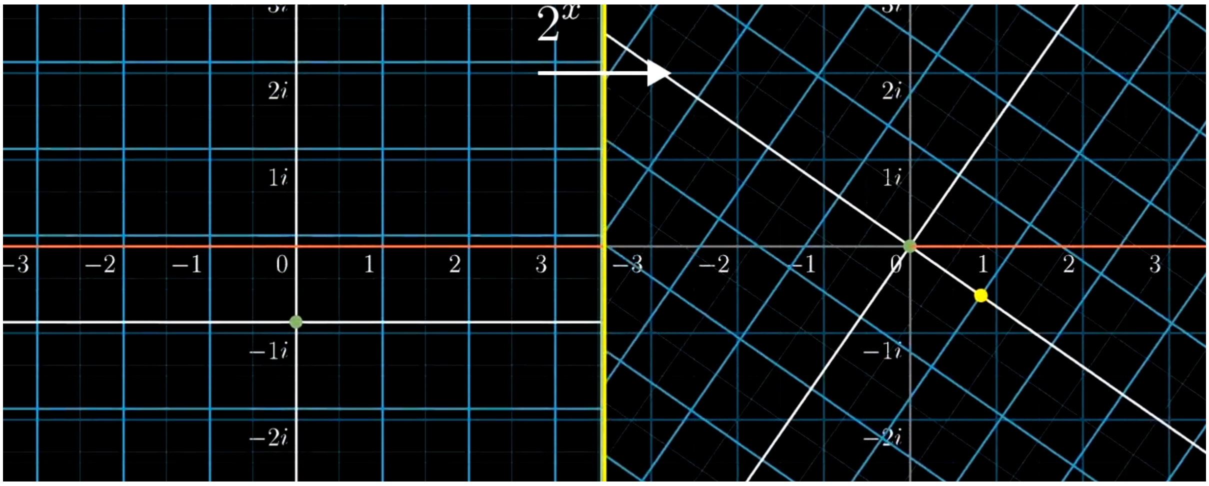
Astonishingly, the additive group of numbers has its correspondent actions in the multiplicative groups. For example, in real number group, if we keep number 0 fixed, we can associate each action in the group with a specific point it's acting on. Following number 1, there is only 1 action could make1 stretched to 3, this action also equals to slide 1 right for 2 units. … …

In this way, every single stretching and squishing action is related to specific sliding action. And this property is also compatible in the frame of complex number. But when it comes to the case of complex number, the counterchange of actions involves rotation. The quintessential example of this is the action associated with point i, 1 unit above 0. It takes 90-degree to rotate 1 to i. So the multiplicative action associated with i is a 90-degree rotation. And if we apply that action twice, the overall effect is to rotate 1 to -1, in other word, i*i = -1. Another example, 2+i. If we rotate 1 up to that point, this equals a rotation of 26.6-degree followed by a stretch by a factor of square root of 5. And in general, every multiplicative action is a combination of action associated with some point on the positive real number line followed by a pure rotation, where pure rotation are associated with points on this circle, the one with radius 1.

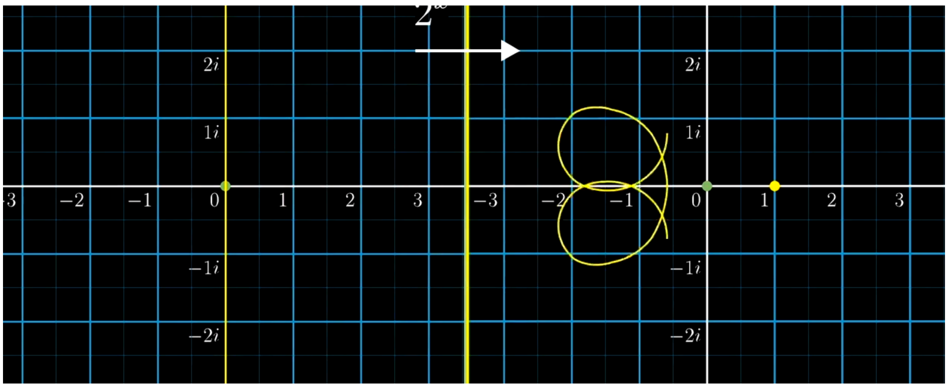


For each group, we can break any action into some purely real number action followed by something that specific to complex numbers, whether that's vertical slides for the additive group, or pure rotation for multiplicative group. And it's reasonable to make sliding affect directly mapped to multiplicative group.

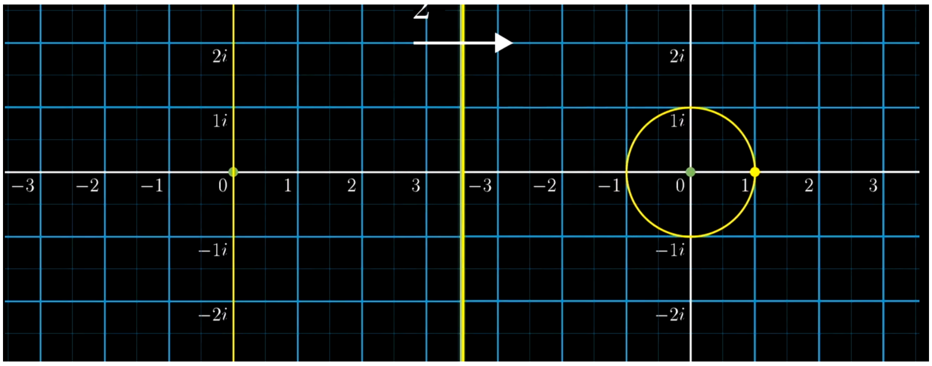
Additive group: Multiplicative group:



Then, if we make the points set on the i-axis of additive group mapped on multiplicative group, a circle would appear:



which becomes:



So what it would mean for an exponential function to map purely vertical slides into pure rotations would be the complex number on i-axis get mapped to the complex numbers on the unit circle (right). In fact, for the exponential function 2^x, a vertical slide of i unit happens to map to the rotation of about 0.693radians; for 5^x, a vertical slide of i unit happens to map to the rotation of 1.609 radians.

What makes number e special is that when e^x maps vertical slides to rotations, the slide of i unit corresponds to exactly 1 radian; he slide of 2i corresponds to 2 radians; the slide of 3i corresponds to 3 radians. And vertical slide of πi corresponds to exactly π radians, half way around the circle. And that is the multiplicative action associated with number -1. And until now, we can fully draw the map of the relationship among these five magical numbers.

Conclusion

Euler's equation is the basis of many scientific subject, it is ubiquitous in mathematics, physics, and engineering. And it is well deserved to be called "our jewel" and "the most remarkable formula in mathematics" (Richard Feynman)

Reference

http://www.math.leidenuniv.nl/~hfinkeln/seminarium/toepassingenEuler.pdf

https://en.wikipedia.org/wiki/List_of_things_named_after_Leonhard_Euler

http://tutorial.math.lamar.edu/Classes/CalcII/TaylorSeries.aspx

https://en.wikipedia.org/wiki/Complex_number#History_in_brief

https://www.zhihu.com/question/22443712/answer/113884840

https://en.wikipedia.org/wiki/Leonhard_Euler

https://www.zhihu.com/question/38428177

http://news.xinhuanet.com/science/2016-03/25/c_135223155.htm

https://zh.wikipedia.org/wiki/%E6%AC%A7%E6%8B%89%E5%85%AC%E5%BC%8F

https://en.wikipedia.org/wiki/Group_theory#Main_classes_of_groups

http://www.bilibili.com/video/av3362960/

https://wenku.baidu.com/view/cf196027ccbff121dd3683c6.html

https://sites.google.com/a/cdsh.ilc.edu.tw/ma/share/shu-xue-fu-hao

https://en.wikipedia.org/wiki/Complex_plane

https://en.wikipedia.org/wiki/Group_theory#History

http://www.bilibili.com/video/av11339177/?from=search&seid=4972939201645427831

http://www.bilibili.com/video/av3362960/?from=search&seid=5454349382632730226