Duke Summer Program

Math and Art of Tessellation

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Math of Universe

Paper 3

July 31, 2017

Introduction

Tessellation is one of the most magnificent parts of geometry. According to many researches, Sumerian who lived in the Euphrates basin created it. In the early stage, tessellations, made by stuck colorful stones, were just simple shapes mainly used to decorate ground. During the second century B.C, tessellation was spread to Rome and Greek where people apply it to decoration of walls and ceilings. As tessellation gradually became popular in Europe, this kind of art developed with great brilliance. After then, M. C. Escher applied some basic patterns to tessellation, accompanying with many mathematic methods, such as reflecting, translating, and rotating, which extraordinarily enriched tessellation. Experiencing centuries of change and participation of various cultures, tessellation contains both plentiful mathematic knowledge and unequaled beauty.

Math of Tessellation

A tessellation is created when one or more shapes is repeated over and over again covering a plane without any gaps or overlaps. Another word for a tessellation is a tiling. Mathematicians use some technical terms when discussing tiling. An edge is the intersection between two bordering tiles; it is often a straight line. A vertex is the point of intersection of three or more bordering tiles.

Tessellation can be generally divided into 2 types: regular tessellations and semi-regular tessellations.

A regular tessellation means a tessellation made up of congruent regular polygons. [Remember: Regular means that the sides and angles of the polygon are all equivalent (i.e., the polygon is both equiangular and equilateral). Congruent means that the polygons that you put together are all the same size and shape.]

A semi-regular tessellation uses a variety of regular polygons. Two necessary properties of it are:

1. It is formed by regular polygons.
2. The arrangement of polygons at every vertex point is identical[1].

- **Exhaustive Attack method**

For these two types of tiling, we can use exhaustive attack method to calculate. What we have already known is that whether we can use one or more regular polygons to tessellate depends on if the requirement that “ the sum of interior angles at every vertex is 360°” can be satisfied.

Since an interior angle of a regular polygon equals

$$(n-2)\times 180°/n=(1/2-1/n)\times 360°$$

To make this kind of tessellation, we should find figure n1, n2, n3 ……, which can satisfy

$$(1/2-1/n1)\times 360°+(1/2-1/n2)\times 360°+(1/2-1/n3)\times 360°+\cdots\cdots=360°$$

In other words $(1/2-1/n1)+(1/2-1/n2)+(1/2-1/n3)+\cdots\cdots=1$

Because we do not know how many figures there are, we can assume that the amount of the figures is m. Thus, we can record (n1, n2, n3,……, nm). So

$$(1/2-1/n1)+(1/2-1/n2)+(1/2-1/nm)+\cdots\cdots=1$$

Then $$m/2-(1/n1+1/n2+\cdots\cdots 1/nm)=1$$

[1] Directly Quote from http://mathforum.org/sum95/suzanne/whattess.html

Namely $$1/n1+1/n2+\cdots\cdots 1/nm= (m-2)/2$$

Since all of the figures are not less than 3,

$$1/n1+1/n2+\cdots\cdots 1/nm \le m/3$$

So $$(m-2)/2 \le m/3$$

$$m \le 6$$

Because, there are at least 3 angles at every vertex, or angles more than 180° will exist. Hence, $m \ge 3$. $\therefore 3 \leqslant m \leqslant 6$.

$$m=3, 4, 5, 6$$

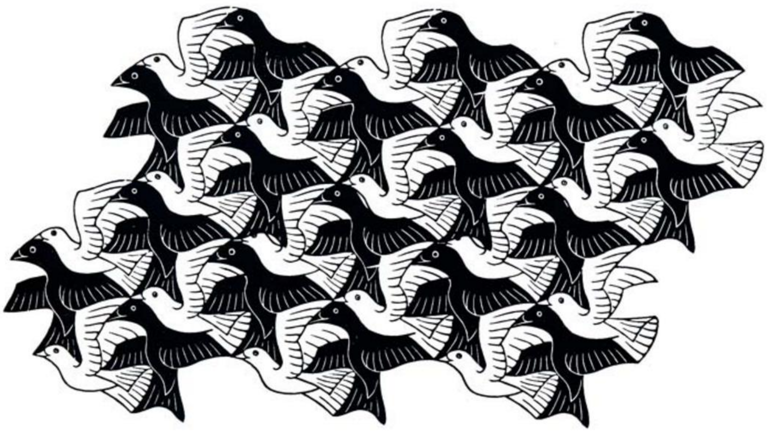
Contra-gradient transformation

In the area of tessellation, M. C. Escher from Netherlands has great achievement. He used some mathematic method, such as reflecting, translating, and rotating, in tessellation and obtain more patterns. He also elaborated these patterns by distorting the basic shapes to render them into animals, birds, and other figures. These distortions had to obey the three, four, or six-fold symmetry of the underlying pattern in order to preserve the tessellation. The effect can be both startling and beautiful.

The first of the examples presented here, titled *Regular Division of the Plane with Birds*, uses a tessellation with triangles. (To see an overlay of the triangle pattern, click on the thumbnail image to expand the large version, and then hover over it with the mouse pointer.)[2]

[2] Directly Quote from http://platonicrealms.com/minitexts/Mathematical-Art-Of-M-C-Escher/

[3]



***Regular Division of the Plane with Birds*; wood engraving, 1949**

Art of Tessellation

Pattern's tessellation can indisputably be classified into a splendid art. Mathematicians search for the knowledge hidden in it, and artists obtain inspirations from mathematics so that they can create work with more visual stimulation, which makes the world of tessellation become more prosperous.

- **Harmony of Tessellation**

Tessellation is well-known all over the world for its constancy and truth of color, diversification of production, and wide range of topics. We can be deeply fascinated by paintings of ancient Rome architecture, mural of ancient Greek, porcelain and lattice of China, and decoration art of Islam. When we observe these work of tiling, it is easy to notice some mathematic concepts like reflection, rotation, symmetry, and association. This identity of art and math efficiently reveals the beauty of harmony.

[3] From http://platonicrealms.com/images/minitexts/escher/regdivbirds_overlay.jpg

In a type of tessellation, there is no one unit can be repeated. Put it in the origin, we can always say it is a new location. Even though there is no rotation or reflection, all the edges can be coincided with the previous ones. Thus, when this kind of tessellation expand to infinity, the amount of its contained different permutation and combination is huge. This is aperiodic tiling. The most famous example of aperiodic tiling is known as Penrose tiling, discovered by Roger Penrose in the 1970s.

● Tessellation of M.C. Escher

[4]The most outstanding point of M.C. Escher' s work is that he gave the object tessellated motion and life.

During 1958-1960, he created a series of <Circle Limit>, using tiling to describe this world as "beautiful limitless and finite plane world".

Besides, we can see how Escher use skills in Geometry like rotation and

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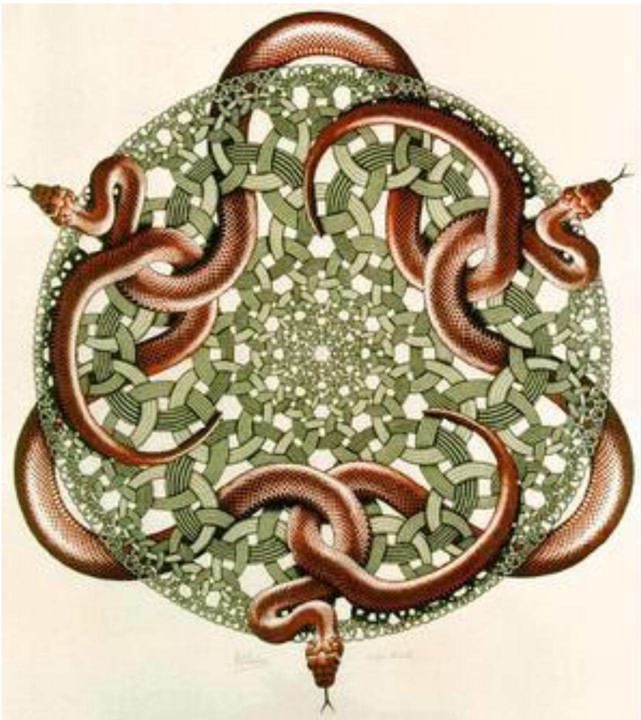
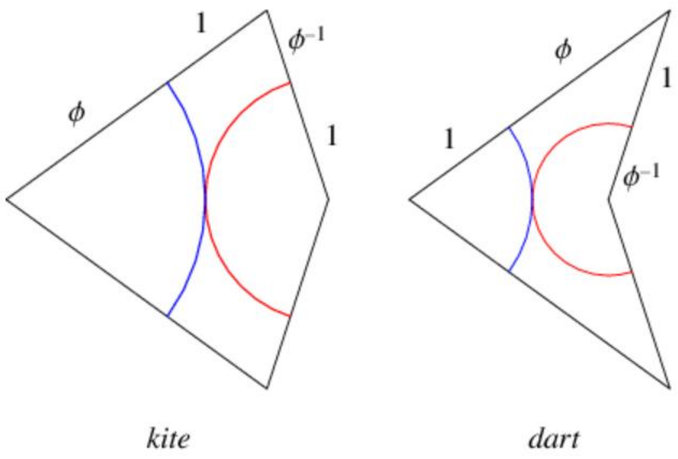
4 From https://upload.wikimedia.org/wikipedia/en/5/55/Escher_Circle_Limit_III.jpg

reflection to let people receive the concept of transform and changing dimensions from <Snakes>. <Snakes> has rotational symmetry of order 3, comprising a single wedge-shaped image repeated three times in a circle. This means that it was printed from three blocks that were rotated on a pin to make three impressions each. Close inspection reveals the central mark left by the pin. The image is printed in three colors: green, brown and black. In several earlier works Escher explored the limits of infinitesimal size and infinite number, for example the Circle Limit series, by actually carrying through the rendering of smaller and smaller figures to the smallest possible sizes. By contrast, in *Snakes*, the infinite diminution of size – and infinite increase in number – is only suggested in the finished work. Nevertheless, the print shows very clearly how [5]this rendering would have been carried out to the limits of human visibility.



[5] From https://upload.wikimedia.org/wikipedia/en/1/16/Escher_Snakes.jpg

● [6]Penrose Tiling



The Penrose tiles are a pair of shapes that tile the plane only non-periodically (when the markings are constrained to match at borders). These two tiles, illustrated above, are called the "kite" and "dart," respectively.

In strict Penrose tiling, the tiles must be placed in such a way that the colored markings agree; in particular, the two tiles may not be combined into a rhombus.[7]

Two additional types of Penrose tiles known as the rhombs (of which there are two varieties: fat and skinny) and the pentacles (or which there are six type) are sometimes also defined that have slightly more complicated matching conditions.

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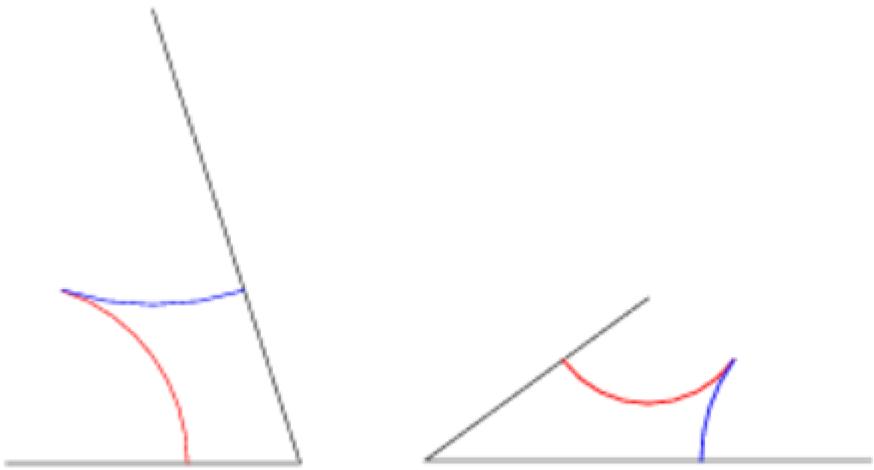
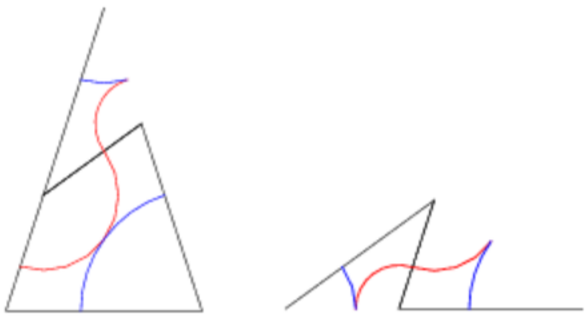
[6] From http://mathworld.wolfram.com/images/eps-gif/PenroseTiles_1000.gif

[7] From Hurd, L. P. "Penrose Tiles." http://library.wolfram.com/infocenter/MathSource/595/.

[8]In 1997, Penrose sued the Kimberly Clark Corporation over their quilted toilet paper, which allegedly resembles a Penrose aperiodic tiling (Mirsky 1997). The suit was apparently settled out of court.



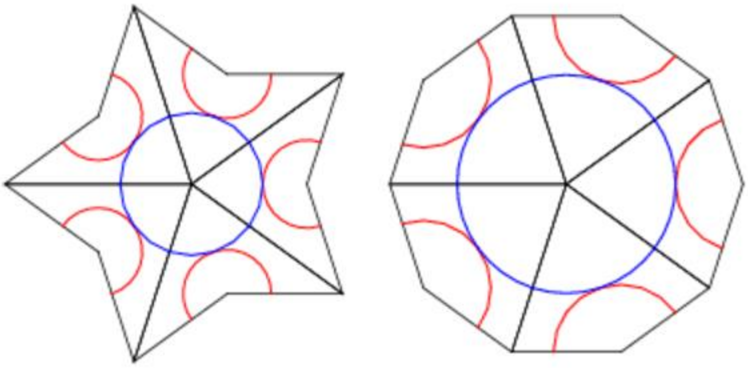
[9]To see how the plane may be tiled non-periodically using the kite and dart, divide the kite into acute and obtuse tiles, shown above.



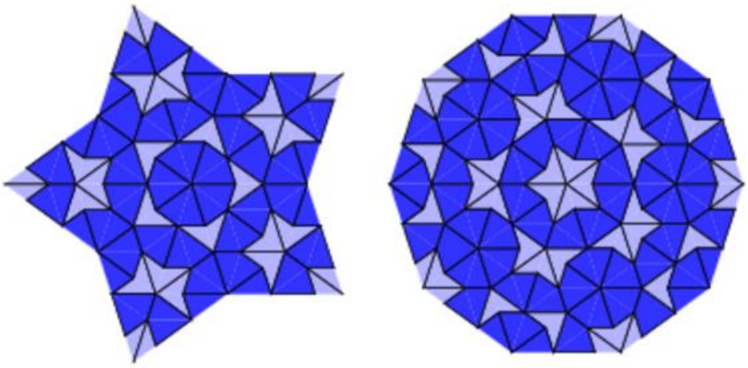
Now define "deflation" and "inflation" operations. The deflation operator takes an acute triangle to the union of two acute triangles and one obtuse, and the obtuse triangle goes to an acute and an obtuse triangle. These operations are illustrated above. Note that the operators do not respect tile boundaries, but do

8 From http://mathworld.wolfram.com/images/eps-gif/PenroseTilesAcuteObtuse_700.gif
9 From http://mathworld.wolfram.com/images/eps-gif/PenroseTilesInflationDeflation_700.gif

[10]respect half-tiles.



[11]When applied to a collection of tiles, the deflation operator leads to a more refined collection. The operators do not respect tile boundaries, but do respect the half tiles defined above. There are two ways to obtain aperiodic tilings with 5-fold symmetry about a single point. They are known as the "star" and "sun" configurations, and are shown above



Higher order versions can then be obtained by deflation. For example, the illustrations above depict the third-order deflations.

John Conway has asked if Penrose tilings are three colorable in such a way that adjacent tiles receive different colors. Sibley and Wagon (2000) proved that

[10] From http://mathworld.wolfram.com/images/eps-gif/PenroseTilesStarSun_1000.gif

[11] From http://mathworld.wolfram.com/images/eps-gif/PenroseTilesStarSun3_1000.gif

tilings by rhombs are three-colorable, and Babilon (2001) proved that tilings by kites and darts are three-colorable. McClure then found an algorithm that appears to three-color tilings by kites and darts, rhombs, and pentacles.

Conclusion

Tessellation occupies many corners in our daily life, such as floors, walls, and abstract art. Thus, tessellation origins from life. When we solve problems about it, we have already had many life experiences. In life of tiling, we observe the internal relationship between math and art, both of which are bright. The mathematics in tessellation broads the bedding of art by sharp beauty and astonishing pure. Also, art inspires people' emotion, enriches our spirit world, and familiarizes math by complete aesthetic.

References

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