Exam 2
Math 353
Summer Term I, 2021 **Name:** ______________________
Friday, June 11, 2021
Time Limit: 75 Minutes

This exam contains 6 pages (including this cover page) and 5 questions. The total number of points on this exam is 60. While this is a closed book, closed notes exam, you are allowed to use your one page review sheet, front and back, written on an 8.5 x 11 inch physical piece of paper in your own handwriting. While your review sheet must have been created with 20th century technology, you may take this exam on a tablet if you like, or on paper.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

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1. (12 points) Consider the differential equation

$$y''(t) + y(t) = \delta(t - \pi) - \delta(t - 3\pi)$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.

(a) Compute the Laplace transform of both sides of the equation and solve for $Y(s)$.

(b) Compute $y(t)$ as the inverse Laplace transform of $Y(s)$.

(c) Plot $y(t)$ for $0 \leq t \leq 4\pi$ and describe the behavior of $y(t)$ for large t.

2. (12 points) Consider the harmonic function u defined in the unit disk $x^2 + y^2 \leq 1$ with boundary conditions $u = f(\theta)$ on the unit circle, where $f(\theta) = \sin(2\theta)$, and θ is the usual polar coordinate.

   Recall that harmonic functions satisfy $u_{xx} + u_{yy} = 0$ in xy coordinates and $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ in polar coordinates.

   (a) Compute $u(r, \theta)$ in the unit disk in polar coordinates.

   (b) Using the fact that $x = r\cos(\theta)$ and $y = r\sin(\theta)$, express u as a function of x and y.

   (c) Verify that this function $u(x, y)$ is harmonic by computing $u_{xx} + u_{yy}$. What is the value of u when $x = 1/10$ and $y = 3/10$?

3. (12 points) Suppose a metal rod represented by the interval $0 \leq x \leq 1$ has an initial temperature of $u(x) = \sin\left(\frac{3\pi x}{2}\right)$ at $t = 0$. Suppose that $u(x,t)$ satisfies the heat equation

$$u_t = u_{xx}$$

for $t \geq 0$, with boundary conditions $u(0,t) = 0$ (left end being kept at a temperature of zero) and $u_x(L,t) = 0$ (right end well insulated).

(a) Compute the temperature $u(x,t)$ of the metal rod for $t \geq 0$.

(b) What is the temperature of the metal rod at $x = 1$ when $t = 10$?

4. (12 points) Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

where

$$a_n = 2\int_0^1 x^3 \cos(n\pi x)\, dx.$$

(a) Is $f(x)$ an even function, an odd function, or neither?

(b) What is the period of $f(x)$?

(c) Graph $f(x)$ for $-3 \le x \le 3$.

(d) What is $f(-5/2)$?

5. (12 points) In this problem, you are NOT ALLOWED to use any sines or cosines. That would just make the problem harder anyway. In this problem, $-\infty < x < \infty$.

   Consider the wave function $u(x,t)$ which is the sum of a left and right traveling wave:

$$u(x,t) = h(x+at) + k(x-at).$$

   (a) Compute u_{tt} and u_{xx} in terms of h and k. Verify that $u(x,t)$ satisfies the wave equation $u_{tt} = a^2 u_{xx}$.

   (b) Compute $u(x,0)$ and $u_t(x,0)$ in terms of the single variable functions h and k.

   (c) Suppose u has initial conditions

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

   for some smooth functions $f(x)$ and $g(x)$. Solve for the corresponding h and k which give these initial conditions. It is okay if your answer has a definite integral in it (which I suggest begins at zero, though it doesn't actually matter).

   (d) Using part (c), derive a formula for $u(x,t)$ in terms of f and g.