Final Exam
Math 353
Summer Term I, 2021
Thursday, June 24
Time Limit: 180 Minutes

Name: Solutions

This exam contains 11 pages (including this cover page) and 10 questions. The total number of points on this exam is 120. While this is a closed book, closed notes exam, you are allowed to use your one page review sheet, front and back, written on an 8.5 x 11 inch physical piece of paper in your own handwriting. While your review sheet must have been created with 20th century technology, you may take this exam on a tablet if you like, or on paper. You may also use your review sheets from the previous two exams.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
10	12	
Total:	120	

1. (12 points) Find the solutions to the following differential equations that are valid in a small neighborhood of the given initial conditions.

   (a) $y' = \frac{x^2}{y}$ where $y = 1$ when $x = 1$.

$y \cdot y' = x^2$

$\frac{1}{2} y^2 = \frac{1}{3} x^3 + C \Rightarrow C = \frac{1}{6}$

$y^2 = \frac{2}{3} x^3 + \frac{1}{3}$

$y = \sqrt{\frac{2}{3} x^3 + \frac{1}{3}}$

   (b) $\frac{dy}{dx} = y^2$ where $y(0) = 1$.

$\frac{dy}{y^2} = dx$

$-\frac{1}{y} = x - C$

$y = \frac{1}{C - x} \Rightarrow C = 1$

$y = \frac{1}{1 - x}$

2. (12 points) Find the general solutions to the following differential equations.

(a) $ty' + 3y = 5t^7$

$\leftarrow$ Multiply by t^2

$$t^3y' + 3t^2y = 5t^9$$

$$(t^3y)' = \left(\tfrac{1}{2}t^{10}\right)'$$

$$t^3y = \tfrac{1}{2}t^{10} + C$$

$$y = \tfrac{1}{2}t^7 + C\cdot t^{-3}$$

(b) $y'' - 10y' + 21y = 4e^{5t}$

$r^2 - 10r + 21 = 0$ for $y = e^{rt}$ for homogeneous equation

$(r-3)(r-7) = 0$

$r = 3, 7$

$y_{\text{Homogeneous}} = C_1e^{3t} + C_2e^{7t}$

Particular Solution: Guess $y = Ae^{5t}$ $\rightarrow$ $25A - 10\cdot 5A + 21A = 4$

$y' = 5Ae^{5t}$ $\qquad -4A = 4$

$y'' = 25Ae^{5t}$ $\qquad A = -1$

General Solution:

$$y = C_1e^{3t} + C_2e^{7t} - e^{5t}$$

3. (12 points) Without just quoting a formula from the book, and using only the definition of the Laplace Transform $\mathscr{L}\{f(t)\}$, prove that

$$\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0).$$

Integration by parts:

$$\boxed{\int_0^\infty u\,dv = uv\Big]_0^\infty - \int_0^\infty v\,du}$$

Definition:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

int. by parts

$$= e^{-st} f'(t)\Big]_0^\infty + \int_0^\infty s\cdot e^{-st} f'(t)\,dt$$

int. by parts

$$= -f'(0) + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= -f'(0) + s\left(e^{-st} f(t)\Big]_0^\infty + \int_0^\infty s e^{-st} f(t)\,dt\right)$$

$$= -f'(0) + s\left(-f(0) + s\int_0^\infty e^{-st} f(t)\,dt\right)$$

$$= -f'(0) - s\cdot f(0) + s^2\mathcal{L}\{f(t)\}$$

4. (12 points) Consider the nonhomogeneous differential equation

$$y'' - 4y' + 4y = 9e^{5t}.$$

(a) What is the general solution to the corresponding homogeneous differential equation (right hand side equal to zero)?

$$r^2 - 4r + 4 = 0$$
$$(r-2)^2 = 0$$
$$r = 2, 2 \rightarrow \text{double root. Hence,}$$

$$y_H(t) = (c_1 + c_2 t)e^{2t}$$

(b) Using the method of undetermined coefficients, find a particular solution to the original nonhomogeneous differential equation.

Guess:

$$\left.\begin{aligned} y &= Ae^{5t} \\ y' &= 5Ae^{5t} \\ y'' &= 25Ae^{5t} \end{aligned}\right\} \quad \begin{aligned} 25A - 4\cdot 5A + 4A &= 9 \\ 9A &= 9 \\ A &= 1 \end{aligned}$$

$$y_p = e^{5t}$$

(c) Find the solution to the original nonhomogeneous differential equation with $y(0) = 2$ and $y'(0) = 7$.

$$y = e^{5t} + (c_1 + c_2 t)e^{2t} \rightarrow 2 = 1 + c_1$$
$$y' = 5e^{5t} + (c_2 + 2c_1 + 2c_2 t)e^{2t} \rightarrow 7 = 5 + c_2 + 2c_1$$

$$c_1 = 1$$
$$c_2 = 0$$

$$y = e^{5t} + e^{2t}$$

5. (12 points) (a) Find the general solution, for $x \neq -1$, to

$$2(x+1)^2 y'' + 3(x+1)y' - y = 0.$$

Let $t = x+1$. Then

$$2t^2 y'' + 3t y' - y = 0$$

Euler type of equation

$$2r(r-1) + 3r - 1 = 0$$

$$2r^2 + r - 1 = 0$$

$$(2r-1)(r+1) = 0$$

$$r = \tfrac{1}{2}, -1$$

$$y = c_1 t^{1/2} + c_2 t^{-1}$$

$$y = c_1 (x+1)^{1/2} + c_2 (x+1)^{-1}$$

(b) Find the particular solution where $y(0) = 5$ and $y'(0) = -2$.

$$y' = \tfrac{1}{2} c_1 (x+1)^{-1/2} - c_2 (x+1)^{-2}$$

$$\boxed{\begin{aligned} 5 &= c_1 + c_2 \\ -2 &= \tfrac{1}{2} c_1 - c_2 \end{aligned}}$$

$$\boxed{\begin{aligned} c_1 &= 2 \\ c_2 &= 3 \end{aligned}}$$

$$y = 2(x+1)^{1/2} + 3(x+1)^{-1}$$

6. (12 points) Consider the differential equation

$$y''(t) - 2y'(t) + 5y(t) = e^{\pi} \cdot \delta(t-\pi) - e^{3\pi} \cdot \delta(t-3\pi)$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.

(a) Compute the Laplace transform of both sides of the equation and solve for $Y(s)$.

$$(s^2 - 2s + 5)\, Y(s) = e^{\pi} \cdot e^{-\pi s} - e^{3\pi} \cdot e^{-3\pi s}$$

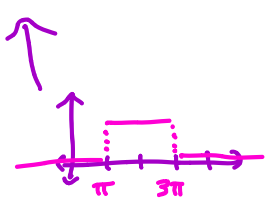
$$Y(s) = \frac{e^{\pi} \cdot e^{-\pi s}}{(s-1)^2 + 2^2} - \frac{e^{3\pi} \cdot e^{-3\pi s}}{(s-1)^2 + 2^2}$$

Note: $\mathcal{L}\left\{\frac{1}{2} e^{t} \sin(2t)\right\} = \frac{1}{(s-1)^2 + 2^2}$

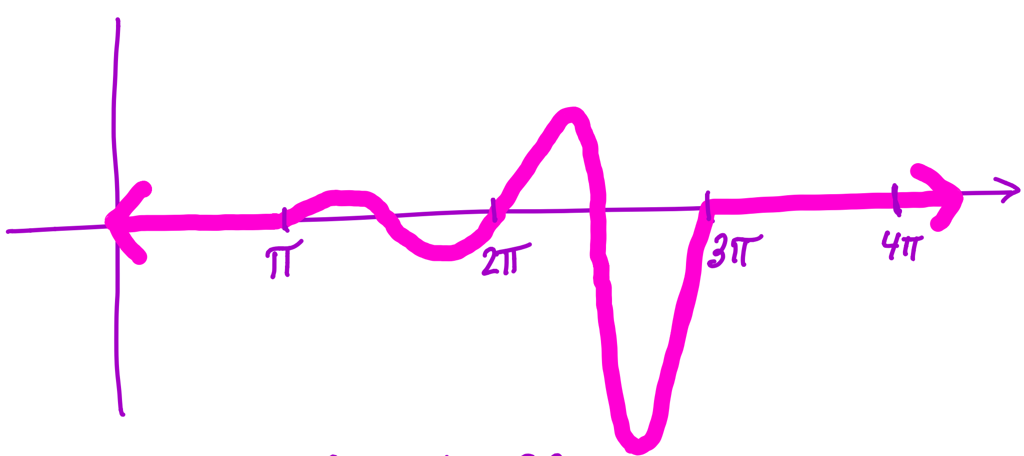
(b) Compute $y(t)$ as the inverse Laplace transform of $Y(s)$.

$$y(t) = \frac{1}{2} e^{\pi} \cdot e^{t-\pi} \sin(2(t-\pi))\, u_{\pi}(t) - \frac{1}{2} e^{3\pi} \cdot e^{t-3\pi} \sin(2(t-3\pi))\, u_{3\pi}(t)$$

$$= \frac{1}{2} e^{t} \cdot \sin(2t) \cdot \left(u_{\pi}(t) - u_{3\pi}(t)\right)$$



(c) Plot $y(t)$ for $0 \le t \le 4\pi$ and describe the behavior of $y(t)$ for large t.



$y(t) = 0$ for $t \geq 3\pi$.

7. (12 points) Suppose a metal rod represented by the interval $0 \le x \le 1$ has an initial temperature of $u(x) = 50 + \cos\left(\frac{5\pi x}{2}\right)$ at $t = 0$. Suppose that $u(x,t)$ satisfies the heat equation

$$u_t = u_{xx}$$

u = 50 solves, so subtract this off at the very beginning and solve the usual Dirichlet boundary condition $u(1,t) \equiv 0$. Then add the 50 back at the very end.

for $t \ge 0$, with boundary conditions $u_x(0,t) = 0$ (left end well insulated) and $u(1,t) = 50$ (right end being kept at a temperature of fifty).

(a) Compute the temperature $u(x,t)$ of the metal rod for $t \ge 0$.

$$u(x,t) = X(x)T(t) \longrightarrow X \cdot T' = X'' T$$

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$$\left.\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X(1) &= 0 \end{aligned}\right\} \longrightarrow X(x) = \cos\left(\left(n - \tfrac{1}{2}\right)\pi x\right)$$

$$\lambda = \left(n - \tfrac{1}{2}\right)^2 \pi^2$$

$$T(t) = e^{-\lambda t}$$

$$\boxed{u_n(x,t) = \cos\left(\left(n-\tfrac{1}{2}\right)\pi x\right) e^{-\left(n-\frac{1}{2}\right)^2 \pi^2 t}}$$

$$\boxed{\begin{aligned} \lambda &= 0 \\ X &= 1 \\ T &= 1 \\ u_0(x,t) &= 1 \end{aligned}}$$

$$u(x,t) = 50 + \cos\left(\frac{5\pi}{2} \cdot x\right) e^{-\frac{25\pi^2}{4} t}$$

(b) What is the temperature of the metal rod at $x = 0$ when $t = 2$?

$$u(0,2) = 50 + \cos(0) \cdot e^{-\frac{25\pi^2}{4} \cdot 2}$$

$$= 50 + e^{-\frac{25\pi^2}{2}}$$

8. (12 points) Let

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x),$$

where

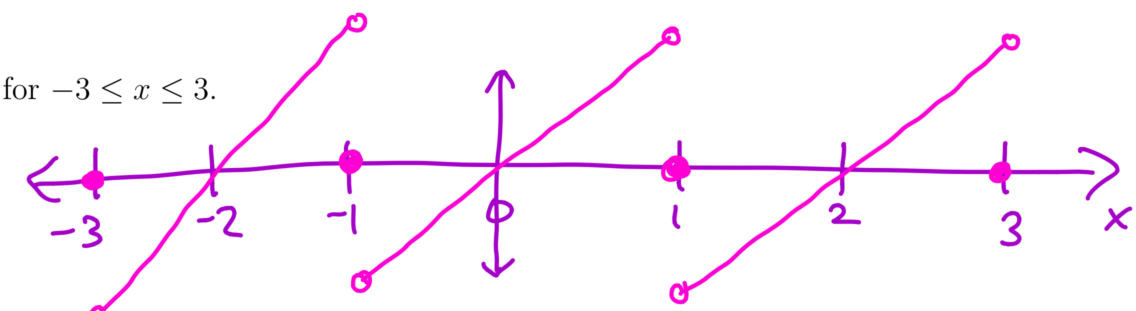
$$a_n = 2\int_0^1 x\sin(n\pi x)\,dx. = \int_{-1}^{1} x \sin(n\pi x)\,dx$$

This is the usual Fourier transform for $f(x) = x$ for $-L \le x \le L$, where $L = 1$.

(a) Is $f(x)$ an even function, an odd function, or neither? What is the period of $f(x)$?

odd 2

(b) Graph $f(x)$ for $-3 \le x \le 3$.



(c) What is $f(7/2)$?

$$f\left(\tfrac{7}{2}\right) = f\left(\tfrac{7}{2} - 4\right) = f\left(-\tfrac{1}{2}\right) = -\tfrac{1}{2}.$$

(d) What is $f(1)$?

$$f(1) = \frac{1}{2}\left(\lim_{x\to 1^+} f(x) + \lim_{x\to 1^-} f(x)\right)$$

$$= \frac{1}{2}\left(-1 + 1\right) = \boxed{0}$$

9. (12 points) A gravitational wave generated from two black holes merging in the Andromeda galaxy is passing through the solar system. Suppose that the distortion of spacetime u is described according to the formula

$$u(x, y, z, t) = f(3x + 4y + 12z - 13t)$$

for some smooth profile function f. Suppose $u(x, y, z, t)$ satisfies the wave equation

$$u_{tt} = v^2(u_{xx} + u_{yy} + u_{zz})$$

for some constant v. Compute the value of v.

$$u_t = -13 \cdot f'(3x+4y+12z-13t)$$
$$u_{tt} = 13^2 \cdot f''(3x+4y+12z-13t)$$

Similarly,
$$u_{xx} = 3^2 \cdot f''(3x+4y+12z-13t)$$
$$u_{yy} = 4^2 \cdot f''(\text{ " }\qquad\text{ " })$$
$$u_{zz} = 12^2 \cdot f''(\text{ " }\qquad\text{ " })$$

$$u_{xx}+u_{yy}+u_{zz} = (9+16+144) f''(3x+4y+12z-13t)$$
$$= 169 \cdot f''(\text{ " }\qquad\text{ " })$$
$$= 13^2 \cdot f''(\text{ " }\qquad\text{ " })$$
$$= u_{tt}.$$

Hence, $v^2 = 1$ so $v = 1$ (or -1).

10. (12 points) Thor and his friends Korg and Miek are trapped in a weird circular room on the planet Sakaar. For our purposes, suppose the room is defined to be the region inside the circle of radius 100 and outside the circle of radius 50, where both circles are centered at the origin.

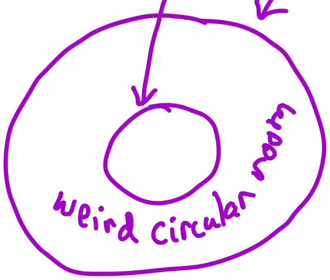
Suppose that the temperature of the inner wall is given in polar coordinates (r, θ) as

$$u(50, \theta) = f(\theta) = 70 + 5\sin(2\theta)$$

and the temperature of the outer wall is also given in polar coordinates as

$$u(100, \theta) = f(\theta) = 70 + 5\sin(2\theta)$$

(a) Is $u(r, \theta) = 70 + 5\sin(2\theta)$ a harmonic function? That is, does it satisfy Laplace's equation in polar coordinates $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$, for all r and θ?



$$\underbrace{u_{rr}}_{0} + \underbrace{\tfrac{1}{r}u_r}_{0} + \underbrace{\tfrac{1}{r^2}u_{\theta\theta}}_{\frac{1}{r^2}\cdot(-20\sin 2\theta)} \neq 0.$$

No! This might have been you guess for part (b), but it is not the answer.

(b) Compute the harmonic function $u(r, \theta)$ with the above boundary conditions which represents the steady state temperature in the room.

$$u(r,\theta) = 70 + \left(C_1 r^2 + C_2 \cdot r^{-2}\right)\sin(2\theta)$$

because separation of variables on $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ gives solutions $\{1, \ln(r), r^n\sin(n\theta), r^n\cos(n\theta), n \neq 0\}$ which includes $\{1, r^2\sin 2\theta, r^{-2}\sin(-2\theta)\}$. See the book for the details. Hence, we just need

$$\left.\begin{aligned} C_1(50)^2 + C_2(50)^{-2} &= 5 \\ C_1(100)^2 + C_2(100)^{-2} &= 5 \end{aligned}\right\} \rightarrow \begin{aligned} (50^2 C_1) + 4\left(\tfrac{C_2}{100^2}\right) &= 5 \\ 4\cdot(50^2 C_1) + \left(\tfrac{C_2}{100^2}\right) &= 5 \end{aligned}$$

$$\downarrow$$

$$50^2 C_1 = 1$$

$$\frac{C_2}{100^2} = 1$$

$$\boxed{u(r,\theta) = 70 + \left[\left(\frac{r}{50}\right)^2 + \left(\frac{100}{r}\right)^2\right]\cdot\sin(2\theta)} \leftarrow$$