Exam 2
Math 353
Summer Term I, 2022
Friday, June 10, 2022
Time Limit: 75 Minutes

Name: Key

This exam contains 7 pages (including this cover page) and 6 questions. The total number of points on this exam is 60. While this is a closed book, closed notes exam, you are allowed to use your one page review sheet for this exam (as well as the one from the first exam) written on an 8.5 x 11 inch physical piece of paper (front and back) in your own handwriting. While your review sheet must have been created with 20th century technology, you may take this exam on a tablet if you like, or on paper.

Show all of your work (on the backs of pages, if necessary) and circle your answers. If you do not completely solve a problem, explain what you do understand about it. If you would like to guess about a problem, clearly say so. This will maximize your partial credit. No collaboration on this exam is allowed. Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	6	
6	6	
Total:	60	

1. (12 points) Consider the differential equation

$$y''(t) + y(t) = \begin{cases} 1, & \text{if } 0 \le t \le 3\pi/2 \\ 0, & \text{if } t > 3\pi/2 \end{cases}$$

with initial conditions $y(0) = 1$ and $y'(0) = 1$.

(a) Compute the Laplace transform of both sides of the equation and solve for $Y(s)$.

$$y''(t) + y(t) = 1 - u_{3\pi/2}(t)$$

$$(s^2 Y(s) - s - 1) + Y(s) = \frac{1}{s} - \frac{1}{s}e^{-\frac{3\pi}{2}s}$$

$$(s^2+1)Y(s) = 1 + s + \frac{1}{s}\left(1 - e^{-\frac{3\pi}{2}s}\right)$$

$$Y(s) = \frac{s+1}{s^2+1} + \frac{1}{s(s^2+1)}\left(1 - e^{-\frac{3\pi}{2}s}\right)$$

(b) Compute $y(t)$ as the inverse Laplace transform of $Y(s)$.

$$\boxed{\frac{1}{s(s^2+1)}} = \frac{a}{s} + \frac{bs+c}{s^2+1} = \frac{(a+b)s^2 + c\cdot s + a}{s(s^2+1)}$$

$$\boxed{\begin{matrix} a = 1 \\ b = -1 \\ c = 0 \end{matrix}}$$

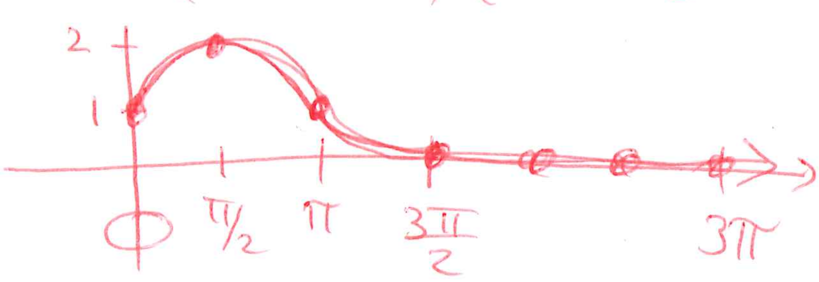
$$= \frac{1}{s} - \frac{s}{s^2+1}$$

$$y(t) = \sin(t) + \cos(t) + (1 - \cos t) - u_{\frac{3\pi}{2}}(t)\left(1 - \cos\left(t - \frac{3\pi}{2}\right)\right)$$

(c) Plot $y(t)$ for $0 \le t \le 3\pi$ and describe the behavior of $y(t)$ for large t.

$$y(t) = 1 + \sin(t) - u_{\frac{3\pi}{2}}(t)(1 + \sin t)$$

$$= (1 + \sin(t))(1 - u_{\frac{3\pi}{2}}(t)) = \begin{cases} 1 + \sin t, & t < \frac{3\pi}{2} \\ 0, & t > \frac{3\pi}{2} \end{cases}$$



2. (12 points) Consider the harmonic function u defined in the unit disk $x^2 + y^2 \leq 1$ with boundary conditions $u = f(\theta)$ on the unit circle, where $f(\theta) = \sin(3\theta)$, and θ is the usual polar coordinate.

Recall that harmonic functions satisfy $u_{xx} + u_{yy} = 0$ in xy coordinates and $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ in polar coordinates.

(a) Compute $u(r, \theta)$ in the unit disk in polar coordinates.

$$u(r,\theta) = r^3 \sin(3\theta)$$

(b) Using the fact that $x = r\cos(\theta)$ and $y = r\sin(\theta)$, express u as a function of x and y. You may find it useful that $\sin(3\theta) = 3\sin(\theta)\cos^2(\theta) - \sin^3(\theta)$.

$$u(x,y) = r^3(3\sin\theta\cos^2\theta - \sin^3\theta)$$
$$= 3x^2y - y^3$$

(c) Verify that this function $u(x, y)$ is harmonic by computing $u_{xx} + u_{yy}$. What is the value of u when $x = 2/10$ and $y = 1/10$?

$$u_{xx} = 6y$$
$$u_{yy} = -6y$$
$$u_{xx} + u_{yy} = 0$$

$$u\left(\tfrac{2}{10}, \tfrac{1}{10}\right) = 3\left(\tfrac{2}{10}\right)^2\left(\tfrac{1}{10}\right) - \left(\tfrac{1}{10}\right)^3$$
$$= \frac{11}{1000}$$
$$= 0.011$$

3. (12 points) Suppose a metal rod represented by the interval $0 \leq x \leq 1$ has an initial temperature of $u(x) = 20 + 10\cos\left(\frac{5\pi x}{2}\right)$ at $t = 0$. Suppose that $u(x,t)$ satisfies the heat equation

$$100\, u_t = u_{xx}$$

for $t \geq 0$, with boundary conditions $u_x(0,t) = 0$ (left end well insulated) and $u(1,t) = 20$ (right end being kept at a temperature of 20).

(a) Compute the temperature $u(x,t)$ of the metal rod for $t \geq 0$.

$$u(x,t) = 20 + 10 \cdot e^{-\left(\frac{5\pi}{2}\right)^2 \frac{t}{100}} \cos\left(\frac{5\pi x}{2}\right)$$

$$= 20 + 10\, e^{-\frac{\pi^2}{16} t} \cos\left(\frac{5\pi x}{2}\right)$$

(b) What is the temperature of the metal rod at $x = 0$ when $t = 1$?

$$u(0,1) = 20 + 10 \cdot e^{-\frac{\pi^2}{16}}$$

4. (12 points) Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x), \quad \leftarrow \text{even, period 2} \quad (L=1)$$

where

$$a_n = 2\int_0^1 x^4 \cos(n\pi x)\, dx. = \int_{-1}^{1} x^4 \cos(n\pi x) dx$$

$$= \frac{1}{L}\int_{-L}^{L} x^4 \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

for $L=1$,

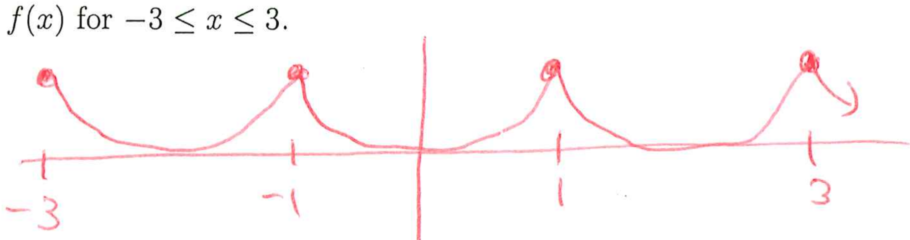
(a) Is $f(x)$ an even function, an odd function, or neither?

even

(b) What is the period of $f(x)$?

2

(c) Graph $f(x)$ for $-3 \le x \le 3$.



(d) What is $f(5/2)$?

$$f\left(\tfrac{5}{2}\right) = f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^4 = \frac{1}{16}$$

5. (6 points) Using separation of variables, or any other method, compute any *bounded* solution $u(x, y, t)$ you like to the 2+1 dimensional wave equation

$$u_{tt} = u_{xx} + u_{yy}$$

which is defined for all x, y, and t and which depends on all three variables (meaning that u_x, u_y, and u_t are nonzero somewhere). Bounded means that the function u stays between two fixed values, like 100 and -100, for all values of x, y, and t.

1. $u(x,y,t) = \sin(3x + 4y + 5t)$

2. $u(x,y,t) = \sin(3x)\sin(4y)\sin(5t)$

6. (6 points) Prove that an even function $(f(-x) = f(x))$ which also satisfies

$$f(5 - x) = -f(5 + x)$$

for all x must be periodic with a period of 20. A picture proof will get some partial credit, but for full credit you must prove this with equations.

$$\begin{aligned}
f(x+20) &= f(5-(-x-15)) \\
&= -f(5+(-x-15)) \\
&= -f(-x-10) \\
&= -f(x+10) \\
&= -f(5-(-x-5)) \\
&= f(5+(-x-5)) \\
&= f(-x) \\
&= f(x)
\end{aligned}$$