Blackjack and Probability

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1 Introduction

Blackjack is an usual game in gambling house and to beat the dealer and make money, people have done lots of research on it. They come up with several basic strategy which is consist of three tables corresponding to the different rules. This paper intend to introduce the rule of blackjack and one basic strategy. The most important part is to tell people how to come up with the basic strategy with excel.

2 Introduction to Blackjack

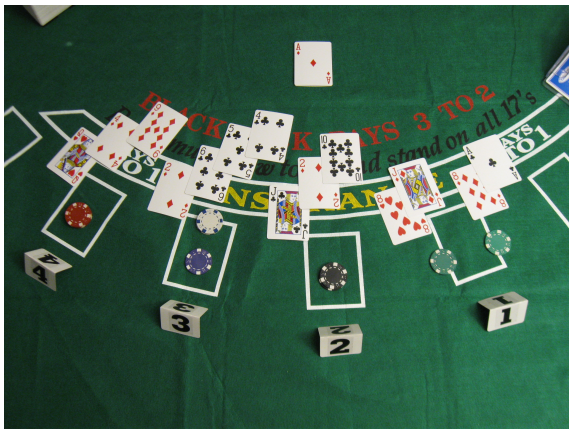
As is known to all, blackjack is an usual game in gambling house, and there are also lots of people play blackjack in their daily life for fun. To some professional gamblers, blackjack is most profitable game in gambling house which means it is easier to win than any other kinds of games. With the best strategy, the probability for gambler to win the game approaches that of the banker most.

2.1 The Process of Blackjack

The game is divided into two sides: dealer and players. The dealer faces five to seven players from behind a semicircular table. The players' object is to win money by creating

card totals that turn out to be higher than the dealer's hand but do not exceed 21 ("busting"/"breaking"), or alternatively by allowing the dealer to take additional cards until he/she busts.

(a) Initial deal



(b) player action

(c) dealer's hand revealed

(d) bets settled

all four pictures are from wikipidea

At the beginning of each round, each player can place his/her bet in the "betting box" at each position in play. The dealer and players get their initial two cards and the dealer's hand receives its first card face up and its second card face down (the hole card), which the dealer peeks at but does not reveal unless it makes the dealer's hand a blackjack(an ace and 10). While the both two cards of each player are usually visible to any other players. Then single cards are dealt to each players clockwise from the dealer's left. On their turn, players must

choose whether to “hit” (take a card), “stand” (end their turn), “double” (double wager, take a single card and finish), “split” (if the two cards have the same value, separate them to make two hands) or “surrender” (give up a half-bet and retire from the game). The more detailed meaning of these player decisions will be listed later in this part. After all boxes have finished playing, the dealer’s hand is resolved by drawing cards until the hand busts or achieves a value of 17 or higher (a dealer total of 17 including an ace, or “soft 17”, must be drawn to in some games and must stand in others, and as the statistic theory shows, the former is more favorable to players).The dealer never doubles, splits, or surrenders.

2.2 The Rule of Blackjack

- Number cards count as their natural value; the jack, queen, and king (also known as “face cards” or “pictures”) count as 10; aces are valued as either 1 or 11 according to the player’s choice.
- If the hand value of player exceeds 21 points, it busts, and all bets on it are immediately forfeit.
- If the dealer busts, all remaining player hands win. If the dealer does not bust, each remaining bet wins if its hand is higher than the dealer’s, and loses if it is lower.
- If a player receives 21 on the 1st and 2nd card it is considered a “natural 21” or “blackjack” and the player is paid out immediately unless dealer also has a natural, in which case the hand ties.
- In the case of a tied score, known as “push” or “standoff”, bets are normally returned without adjustment.
- A blackjack beats any hand that is not a blackjack, even one with a value of 21.
- An outcome of blackjack vs blackjack results in a push. Wins are paid out at 1:1, or equal to the wager, except for winning blackjacks, which are traditionally paid at 3:2

(meaning the player receives three dollars for every two bet).

- Blackjack games almost always provide a side bet called insurance, which may be played when dealer's upcard is an ace. Additional side bets, such as "Dealer Match" which pays when the player's cards match the dealer's up card, are sometimes available.

2.3 Player Decisions

- Hit: Take another card from the dealer.

- Stand: Take no more cards, also known as "stand pat", "stick", or "stay".

- Double down: The player is allowed to increase the initial bet by up to 100 % in exchange for committing to stand after receiving exactly one more card. The additional bet is placed in the betting box next to the original bet. Some games do not permit the player to increase the bet by amounts other than 100 %. Non-controlling players may double their wager or decline to do so, but they are bound by the controlling player's decision to take only one card.

- Split: If the first two cards of a hand have the same value, the player can split them into two hands, by moving a second bet equal to the first into an area outside the betting box. The dealer separates the two cards and draws an additional card on each, placing one bet with each hand. The player then plays out the two separate hands in turn; except for a few restrictions, the hands are treated as independent new hands, with the player winning or losing their wager separately for each hand. Occasionally, in the case of ten-valued cards, some casinos allow splitting only when the cards have the identical ranks; for instance, a hand of 10-10 may be split, but not one of 10-king. However, usually all 10-value cards are treated the same. Doubling and further splitting of post-split hands may be restricted, and blackjacks after a split are counted as non-blackjack 21 when comparing against the dealer's hand. Hitting split aces is usually

not allowed. Non-controlling players may follow the controlling player by putting down an additional bet or decline to do so, instead associating their existing wager with one of the two post-split hands. In that case they must choose which hand to play behind before the second cards are drawn. Some casinos do not give non-controlling players this option, and require that the wager of a player not electing to split remains with the first of the two post-split hands.

- Surrender (only available as first decision of a hand): Some games offer the option to "surrender", usually in hole-card games and directly after the dealer has checked for blackjack (but see below for variations). When the player surrenders, the house takes half the player's bet and returns the other half to the player; this terminates the player's interest in the hand. [1]

3 How to Play Blackjack

As we can see, blackjack is not that complex. To win the game and make more money, all we need to do is remember the basic strategy, which is consist of three tables.

3.1 The Basic Strategy

Each blackjack game has a basic strategy, which is playing a hand of any total value against any dealer's up-card, which loses the least money to the house in the long term.

An example of basic strategy is shown in the table below, and includes the following parameters:[2]

- Four to eight decks
- The dealer stands on a soft 17
- A double is allowed after a split

[1] from wikipedia

[2] "4-Deck to 8-Deck Blackjack Strategy - Wizard of Odds". Wizard of Odds Consulting, Inc. Retrieved 30 April 2014.

- Only original bets are lost on dealer blackjack

Player hand	Dealer's face-up card									
	2	3	4	5	6	7	8	9	10	A
Hard totals (excluding pairs)										
17–20	S	S	S	S	S	S	S	S	S	S
16	S	S	S	S	S	H	H	SU	SU	SU
15	S	S	S	S	S	H	H	H	SU	H
13–14	S	S	S	S	S	H	H	H	H	H
12	H	H	S	S	S	H	H	H	H	H
11	Dh	Dh	Dh	Dh	Dh	Dh	Dh	Dh	Dh	H
10	Dh	Dh	Dh	Dh	Dh	Dh	Dh	Dh	H	H
9	H	Dh	Dh	Dh	Dh	H	H	H	H	H
5–8	H	H	H	H	H	H	H	H	H	H
Soft totals										
	2	3	4	5	6	7	8	9	10	A
A,8–A,9	S	S	S	S	S	S	S	S	S	S
A,7	S	Ds	Ds	Ds	Ds	S	S	H	H	H
A,6	H	Dh	Dh	Dh	Dh	H	H	H	H	H
A,4–A,5	H	H	Dh	Dh	Dh	H	H	H	H	H
A,2–A,3	H	H	H	Dh	Dh	H	H	H	H	H
Pairs										
	2	3	4	5	6	7	8	9	10	A
A,A	SP	SP	SP	SP	SP	SP	SP	SP	SP	SP
10,10	S	S	S	S	S	S	S	S	S	S
9,9	SP	SP	SP	SP	SP	S	SP	SP	S	S
8,8	SP	SP	SP	SP	SP	SP	SP	SP	SP	SP
7,7	SP	SP	SP	SP	SP	SP	H	H	H	H
6,6	SP	SP	SP	SP	SP	H	H	H	H	H
5,5	Dh	Dh	Dh	Dh	Dh	Dh	Dh	Dh	H	H
4,4	H	H	H	SP	SP	H	H	H	H	H
2,2–3,3	SP	SP	SP	SP	SP	SP	H	H	H	H

from wikipedia

Key:

- S = Stand
- H = Hit
- Dh = Double (if not allowed, then hit)
- Ds = Double (if not allowed, then stand)
- SP = Split
- SU = Surrender (if not allowed, then hit)

3.2 How to Make the Table of Basic Strategy

3.2.1 Probability of Dealer

To begin with, we will deal with the probablity of dealer. The outcome of dealer's hand is bust and 17∼21. That means we need to compute the probablity of each outcome when the face up card of the dealer is 2∼10 and ace. This can clearly form a table called "dealer". We will use the method of recursion. For instance, if we have already known the probabilities that dealer's outcome is 17 when the card in hand is 4∼12 and soft 13. Then we can get the probability that dealer's outcome is 17 when the hand is 2.

To achieve this, we need two auxiliary tables. Let first one be "hard". Its column is consist of 2∼31 without ace and the row is the same as "dealer". Let the other one be "soft". Its column is consist of 12∼31 and the row is also the same. We can easily initialize the two table as following:

outcome	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
bust																		
17																1		
18																	1	
19																		1
20																		
21																		

hard-1

20	21	22	23	24	25	26	27	28	29	30	31
		1	1	1	1	1	1	1	1	1	1
1											
	1										

hard-2

soft																				
12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
										0	0	0	0	0	0	0	0	0	0
					1					0	0	0	0	0	1	0	0	0	0
						1				0	0	0	0	0	0	1	0	0	0
							1			0	0	0	0	0	0	0	1	0	0
								1		0	0	0	0	0	0	0	0	1	0
									1	0	0	0	0	0	0	0	0	0	1

soft

Let $P_y(x)$ be the probability that the dealer's outcome is y when the hand is x.

In "hard", it is easy to understand that when the hand is 17, the probability that the outcome is 17 is 1, and when the hand is 22 or even higher, the probability for the dealer to bust is 1. In "soft", the column 22∼31 is just the same as the column 12∼21 in "hard". Apply the formula (1) to every blank in "hard" column 2∼16.

$$P_y^{hard}(x) = \frac{1}{13}(\sum_{n=2}^{9} P_y^{hard}(x+n) + 4P_y^{hard}(x+10) + P_y^{soft}(x+11)) \tag{1}$$

Because there are four cards standing for 10, so the coefficient of $P_y^{hard}(x+10)$is 4.

As for the blank in "soft" column 12∼16, we can apply the formula (2):

$$P_y^{soft}(x) = \frac{1}{13}(\sum_{n=1}^{9} P_y^{soft}(x+n) + 4P_y^{soft}(x+10)) \tag{2}$$

Now, we have the two auxiliary tables ready, so we can start to deal with the "dealer". Because it is impossible to have a blackjack when the face up card is 2∼9, so the "dealer" column 2∼9 is just the same as the "hard" column 2∼9. As for the column 10, we need to

exclude the possibility that the dealer gets an ace, so the formula will be

$$P_y^{dealer}(10) = \frac{1}{12}(\sum_{x=12}^{19} P_y^{hard}(x) + 4P_y^{hard}(20)) \tag{3}$$

As for the column ace, we need to exclude the possibility that the dealer gets 10, so the formula will be

$$P_y^{dealer}(ace) = \frac{1}{9}\sum_{x=12}^{20} P_y^{soft}(x) \tag{4}$$

So, we get the "dealer" table:

outcome	2	3	4	5	6	7	8	9	10	ace
bust	0.35360814	0.37387489	0.39446845	0.41640367	0.42315049	0.26231241	0.24474124	0.22842516	0.22978483	0.16652461
17	0.13980914	0.13503399	0.13048974	0.12225129	0.16543818	0.36856619	0.12856654	0.11999544	0.1207097	0.1889173
18	0.13490735	0.13048233	0.12593807	0.12225129	0.10626658	0.13779696	0.35933578	0.11999544	0.1207097	0.1889173
19	0.12965543	0.12558054	0.12138641	0.11769962	0.10626658	0.07862537	0.12856654	0.35076467	0.1207097	0.1889173
20	0.12402646	0.12032862	0.11648462	0.11314796	0.10171492	0.07862537	0.06939495	0.11999544	0.3707097	0.1889173
21	0.11799348	0.11469964	0.11123271	0.10824617	0.09716326	0.0740737	0.06939495	0.06082384	0.03737637	0.07780619

dealer

3.2.2 Expected Value when Standing

In this part, we will discuss the probability for player to win when they choose to stand. We will also need two tables. The first one is "hard-S" whose column is consist of 2∼10 and ace and row is consist of 4∼31. Let each blank be $P_x^{hard-S}(y)$ which means the expected value of player when the dealer's face up card is x, the hand of player is y and player chooses to stand. For instance, when the hand of player is 4 and face up card is 2, the only way for player to win is that the dealer busts. So we can get $P_2^{hard-S}(4) = P_{bust}^{dealer}(2)$. And the same thing will happen when $y = 5 \sim 16$. And when $y = 17 \sim 21$, we will have the formula below

$$P_x^{hard-S}(y) = P_{bust}^{dealer}(x) + \sum_{z<y} P_z^{dealer}(x) - \sum_{z>y} P_z^{dealer}(x) \tag{5}$$

So it is easy to see that the value can be negative. As for $y > 21$, the player just busts so he/she is bound to lose and $P_x^{hard-S}(y) = -1$.So we get the "hard-S" as below

hard-S	2	3	4	5	6	7	8	9	10	ace
4	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
5	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
6	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
7	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
8	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
9	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
10	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
11	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
12	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
13	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
14	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
15	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
16	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
17	−0. 1529746	−0. 1172162	−0. 0805734	−0. 0449414	0. 01173916	−0. 106809	−0. 381951	−0. 4231542	−0. 4197206	−0. 4780335
18	0. 1217419	0. 14830007	0. 17585444	0. 19956119	0. 28344392	0. 39955417	0. 10595135	−0. 1831634	−0. 1783012	−0. 1001989
19	0. 38630469	0. 40436294	0. 42317892	0. 4395121	0. 49597707	0. 6159765	0. 59385367	0. 28759676	0. 06311817	0. 27763572
20	0. 63998658	0. 65027209	0. 66104996	0. 67035969	0. 70395857	0. 77322723	0. 79181516	0. 75835687	0. 55453757	0. 65547032
21	0. 88200652	0. 88530036	0. 88876729	0. 89175383	0. 90283674	0. 9259263	0. 93060505	0. 93917616	0. 96262363	0. 92219381
22	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
23	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
24	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
25	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
26	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
27	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
28	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
29	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
30	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
31	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1

hard-S

The other table is soft-S. The row of "soft-S" is consist of 12∼31 and the column is just the same as "hard-S". It is clear to everyone that

$$P_x^{soft-S}(y) = \begin{cases} P_x^{hard-S}(y), & y = 12 \sim 21 \\ P_x^{hard-S}(y-10), & y = 22 \sim 31. \end{cases} \tag{6}$$

So we get "soft-S" as below

soft-S	2	3	4	5	6	7	8	9	10	ace
12	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
13	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
14	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
15	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
16	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
17	−0. 1529746	−0. 1172162	−0. 0805734	−0. 0449414	0. 01173916	−0. 106809	−0. 381951	−0. 4231542	−0. 4197206	−0. 4780335
18	0. 1217419	0. 14830007	0. 17585444	0. 19956119	0. 28344392	0. 39955417	0. 10595135	−0. 1831634	−0. 1783012	−0. 1001989
19	0. 38630469	0. 40436294	0. 42317892	0. 4395121	0. 49597707	0. 6159765	0. 59385367	0. 28759676	0. 06311817	0. 27763572
20	0. 63998658	0. 65027209	0. 66104996	0. 67035969	0. 70395857	0. 77322723	0. 79181516	0. 75835687	0. 55453757	0. 65547032
21	0. 88200652	0. 88530036	0. 88876729	0. 89175383	0. 90283674	0. 9259263	0. 93060505	0. 93917616	0. 96262363	0. 92219381
22	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
23	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
24	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
25	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
26	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4753752	−0. 5105175	−0. 5431497	−0. 5404303	−0. 6669508
27	−0. 1529746	−0. 1172162	−0. 0805734	−0. 0449414	0. 01173916	−0. 106809	−0. 381951	−0. 4231542	−0. 4197206	−0. 4780335
28	0. 1217419	0. 14830007	0. 17585444	0. 19956119	0. 28344392	0. 39955417	0. 10595135	−0. 1831634	−0. 1783012	−0. 1001989
29	0. 38630469	0. 40436294	0. 42317892	0. 4395121	0. 49597707	0. 6159765	0. 59385367	0. 28759676	0. 06311817	0. 27763572
30	0. 63998658	0. 65027209	0. 66104996	0. 67035969	0. 70395857	0. 77322723	0. 79181516	0. 75835687	0. 55453757	0. 65547032
31	0. 88200652	0. 88530036	0. 88876729	0. 89175383	0. 90283674	0. 9259263	0. 93060505	0. 93917616	0. 96262363	0. 92219381

soft-S

3.2.3 Expected Value when Hitting

In this part, we will discuss similarly to the former part. We will give two table which are "hard-H" and "soft-H". Let each blank be $P_x^{hard-H/soft-H}(y)$ which means the expected value of player when the dealer's face up card is x, the hand of player is y and player chooses to hit. And we will also have another two table which are "hard-hs" and "soft-hs". They are defined as below

$$P_x^{hard-hs}(y) = max(P_x^{hard-H}(y), P_x^{hard-S}(y)) \tag{7}$$

$$P_x^{soft-hs}(y) = max(P_x^{soft-H}(y), P_x^{soft-S}(y)) \tag{8}$$

Because the player have to make the choice with higher expected value.

Then we get the formula

$$P_x^{hard-H}(y) = \begin{cases} \frac{1}{13}(\sum_{n=2}^{9} P_x^{hard-hs}(y+n) + 4P_x^{hard-hs}(y+10) + P_x^{soft-hs}(y+11)), & y = 4 \sim 20 \\ -1, & y = 21 \sim 31. \end{cases} \tag{9}$$

and

$$P_x^{soft-H}(y) = \begin{cases} \frac{1}{13}(\sum_{n=1}^{9} P_x^{soft-hs}(y+n) + 4P_x^{soft-hs}(y+10)), & y = 12 \sim 21 \\ P_x^{hard-hs}(y-10), & y = 22 \sim 31. \end{cases} \tag{10}$$

So we get the four tables as below

hard-H	2	3	4	5	6	7	8	9	10	ace
4	−0.1149133	−0.0826133	−0.0493674	−0.0123799	0.01113042	−0.0882792	−0.1593342	−0.2406662	−0.2891979	−0.253077
5	−0.1282156	−0.0953102	−0.0614795	−0.023979	−0.0011863	−0.1194474	−0.1880933	−0.2666151	−0.3134116	−0.2785746
6	−0.1407591	−0.1072911	−0.0729171	−0.034916	−0.0130058	−0.1519327	−0.2172419	−0.2926407	−0.3377494	−0.3041466
7	−0.1091834	−0.076583	−0.0430218	−0.0072714	0.02918534	−0.0688078	−0.2106048	−0.2853654	−0.3190548	−0.3100717
8	−0.0217982	0.00800526	0.03878447	0.07080464	0.11496015	0.08220744	−0.0598983	−0.2101863	−0.2493751	−0.1970288
9	0.07444604	0.1012647	0.12898088	0.15803186	0.19601884	0.17186786	0.09837622	−0.0521781	−0.152953	−0.0656808
10	0.18249999	0.20608798	0.23047012	0.25625855	0.28779508	0.25690874	0.19795371	0.11652959	0.02530852	0.08144971
11	0.23835075	0.26032527	0.28302028	0.30734951	0.33369005	0.29214699	0.22998215	0.15825712	0.11948223	0.14300128
12	−0.25339	−0.2336909	−0.2135366	−0.1932712	−0.1705262	−0.2128477	−0.2715748	−0.3400133	−0.381043	−0.3505403
13	−0.3077912	−0.2912101	−0.274224	−0.2573333	−0.2356263	−0.2690729	−0.3236052	−0.3871552	−0.4252542	−0.3969303
14	−0.3621925	−0.3487293	−0.3349115	−0.3213954	−0.3007264	−0.321282	−0.3719191	−0.4309298	−0.4663075	−0.4400067
15	−0.4165937	−0.4062485	−0.3955989	−0.3854575	−0.3658264	−0.3697618	−0.416782	−0.4715777	−0.5044284	−0.4800062
16	−0.470995	−0.4637678	−0.4562864	−0.4495196	−0.4309265	−0.4147788	−0.4584404	−0.5093221	−0.5398263	−0.5171487
17	−0.5361508	−0.5316742	−0.5270115	−0.5229856	−0.5087526	−0.4834858	−0.5059827	−0.5536949	−0.5844632	−0.5572999
18	−0.6224386	−0.620005	−0.6174618	−0.6152596	−0.607479	−0.5911438	−0.5910559	−0.6165285	−0.6476708	−0.6265154
19	−0.7290775	−0.7280329	−0.7269371	−0.7259913	−0.7225542	−0.7154497	−0.71366	−0.7155744	−0.7294491	−0.7247951
20	−0.8552303	−0.8549769	−0.8547102	−0.8544805	−0.8536279	−0.8518518	−0.8514919	−0.8508326	−0.849029	−0.8521389
21	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
22	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
23	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
24	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
25	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
26	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
27	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
28	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
29	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
30	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
31	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1

hard-H

soft-H	2	3	4	5	6	7	8	9	10	ace
12	0.08183622	0.10350705	0.12659563	0.15648238	0.18595361	0.16547293	0.09511502	6.5791E-05	-0.0700024	-0.0204779
13	0.04663613	0.07411881	0.10247715	0.13336274	0.16169271	0.1223857	0.05405707	-0.0376947	-0.1048514	-0.057308
14	0.02239186	0.05080674	0.08008141	0.1118945	0.13916473	0.07950749	0.01327722	-0.0751632	-0.1394668	-0.0938743
15	-0.0001207	0.02915981	0.05928538	0.0919597	0.11824589	0.03702828	-0.0270548	-0.1121888	-0.1737042	-0.1300265
16	-0.0210252	0.0090591	0.03997477	0.07344882	0.09882126	-0.0048902	-0.0667948	-0.1486435	-0.2074411	-0.1656372
17	-0.000491	0.02897528	0.05932628	0.09118908	0.12805214	0.05382346	-0.0729154	-0.1497869	-0.196867	-0.1795694
18	0.06290507	0.09024828	0.11850192	0.14761275	0.19075324	0.1706765	0.03967744	-0.1007443	-0.1438081	-0.0929355
19	0.12395802	0.14933971	0.17557681	0.20298603	0.23979935	0.22062011	0.15227029	0.00789264	-0.088096	-0.0057429
20	0.18249999	0.20608798	0.23047012	0.25625855	0.28779508	0.25690874	0.19795371	0.11652959	0.02530852	0.08144971
21	0.23835075	0.26032527	0.28302028	0.30734951	0.33369005	0.29214699	0.22998215	0.15825712	0.11948223	0.14300128
22	-0.25339	-0.2336909	-0.2135366	-0.1932712	-0.1705262	-0.2128477	-0.2715748	-0.3400133	-0.381043	-0.3505403
23	-0.3077912	-0.2912101	-0.274224	-0.2573333	-0.2356263	-0.2690729	-0.3236052	-0.3871552	-0.4252542	-0.3969303
24	-0.3621925	-0.3487293	-0.3349115	-0.3213954	-0.3007264	-0.321282	-0.3719191	-0.4309298	-0.4663075	-0.4400067
25	-0.4165937	-0.4062485	-0.3955989	-0.3854575	-0.3658264	-0.3697618	-0.416782	-0.4715777	-0.5044284	-0.4800062
26	-0.470995	-0.4637678	-0.4562864	-0.4495196	-0.4309265	-0.4147788	-0.4584404	-0.5093221	-0.5398263	-0.5171487
27	-0.5361508	-0.5316742	-0.5270115	-0.5229856	-0.5087526	-0.4834858	-0.5059827	-0.5536949	-0.5844632	-0.5572999
28	-0.6224386	-0.620005	-0.6174618	-0.6152596	-0.607479	-0.5911438	-0.5910559	-0.6165285	-0.6476708	-0.6265154
29	-0.7290775	-0.7280329	-0.7269371	-0.7259913	-0.7225542	-0.7154497	-0.71366	-0.7155744	-0.7294491	-0.7247951
30	-0.8552303	-0.8549769	-0.8547102	-0.8544805	-0.8536279	-0.8518518	-0.8514919	-0.8508326	-0.849029	-0.8521389
31	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

soft-H

hard-hs	2	3	4	5	6	7	8	9	10	ace
4	-0.1149133	-0.0826133	-0.0493674	-0.0123799	0.01113042	-0.0882792	-0.1593342	-0.2406662	-0.2891979	-0.253077
5	-0.1282156	-0.0953102	-0.0614795	-0.023979	-0.0011863	-0.1194474	-0.1880933	-0.2666151	-0.3134116	-0.2785746
6	-0.1407591	-0.1072911	-0.0729171	-0.034916	-0.0130058	-0.1519327	-0.2172419	-0.2926407	-0.3377494	-0.3041466
7	-0.1091834	-0.076583	-0.0430218	-0.0072714	0.02918534	-0.0688078	-0.2106048	-0.2853654	-0.3190548	-0.3100717
8	-0.0217982	0.00800526	0.03878447	0.07080464	0.11496015	0.08220744	-0.0598983	-0.2101863	-0.2493751	-0.1970288
9	0.07444604	0.1012647	0.12898088	0.15803186	0.19601884	0.17186786	0.09837622	-0.0521781	-0.152953	-0.0656808
10	0.18249999	0.20608798	0.23047012	0.25625855	0.28779508	0.25690874	0.19795371	0.11652959	0.02530852	0.08144971
11	0.23835075	0.26032527	0.28302028	0.30734951	0.33369005	0.29214699	0.22998215	0.15825712	0.11948223	0.14300128
12	-0.25339	-0.2336909	-0.2110631	-0.1671927	-0.153699	-0.2128477	-0.2715748	-0.3400133	-0.381043	-0.3505403
13	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.2690729	-0.3236052	-0.3871552	-0.4252542	-0.3969303
14	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.321282	-0.3719191	-0.4309298	-0.4663075	-0.4400067
15	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.3697618	-0.416782	-0.4715777	-0.5044284	-0.4800062
16	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.4147788	-0.4584404	-0.5093221	-0.5398263	-0.5171487
17	-0.1529746	-0.1172162	-0.0805734	-0.0449414	0.01173916	-0.106809	-0.381951	-0.4231542	-0.4197206	-0.4780335
18	0.1217419	0.14830007	0.17585444	0.19956119	0.28344392	0.39955417	0.10595135	-0.1831634	-0.1783012	-0.1001989
19	0.38630469	0.40436294	0.42317892	0.4395121	0.49597707	0.6159765	0.59385367	0.28759676	0.06311817	0.27763572
20	0.63998658	0.65027209	0.66104996	0.67035969	0.70395857	0.77322723	0.79181516	0.75835687	0.55453757	0.65547032
21	0.88200652	0.88530036	0.88876729	0.89175383	0.90283674	0.9259263	0.93060505	0.93917616	0.96262363	0.92219381
22	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
23	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
24	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
25	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
26	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
27	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
28	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
29	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
30	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
31	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

hard-hs

soft-hs	2	3	4	5	6	7	8	9	10	ace
12	0.08183622	0.10350705	0.12659563	0.15648238	0.18595361	0.16547293	0.09511502	6.5791E-05	-0.0700024	-0.0204779
13	0.04663613	0.07411881	0.10247715	0.13336274	0.16169271	0.1223857	0.05405707	-0.0376947	-0.1048514	-0.057308
14	0.02239186	0.05080674	0.08008141	0.1118945	0.13916473	0.07950749	0.01327722	-0.0751632	-0.1394668	-0.0938743
15	-0.0001207	0.02915981	0.05928538	0.0919597	0.11824589	0.03702828	-0.0270548	-0.1121888	-0.1737042	-0.1300265
16	-0.0210252	0.0090591	0.03997477	0.07344882	0.09882126	-0.0048902	-0.0667948	-0.1486435	-0.2074411	-0.1656372
17	-0.000491	0.02897528	0.05932628	0.09118908	0.12805214	0.05382346	-0.0729154	-0.1497869	-0.196867	-0.1795694
18	0.1217419	0.14830007	0.17585444	0.19956119	0.28344392	0.39955417	0.10595135	-0.1007443	-0.1438081	-0.0929355
19	0.38630469	0.40436294	0.42317892	0.4395121	0.49597707	0.6159765	0.59385367	0.28759676	0.06311817	0.27763572
20	0.63998658	0.65027209	0.66104996	0.67035969	0.70395857	0.77322723	0.79181516	0.75835687	0.55453757	0.65547032
21	0.88200652	0.88530036	0.88876729	0.89175383	0.90283674	0.9259263	0.93060505	0.93917616	0.96262363	0.92219381
22	-0.25339	-0.2336909	-0.2110631	-0.1671927	-0.153699	-0.2128477	-0.2715748	-0.3400133	-0.381043	-0.3505403
23	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.2690729	-0.3236052	-0.3871552	-0.4252542	-0.3969303
24	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.321282	-0.3719191	-0.4309298	-0.4663075	-0.4400067
25	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.3697618	-0.416782	-0.4715777	-0.5044284	-0.4800062
26	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.4147788	-0.4584404	-0.5093221	-0.5398263	-0.5171487
27	-0.1529746	-0.1172162	-0.0805734	-0.0449414	0.01173916	-0.106809	-0.381951	-0.4231542	-0.4197206	-0.4780335
28	0.1217419	0.14830007	0.17585444	0.19956119	0.28344392	0.39955417	0.10595135	-0.1831634	-0.1783012	-0.1001989
29	0.38630469	0.40436294	0.42317892	0.4395121	0.49597707	0.6159765	0.59385367	0.28759676	0.06311817	0.27763572
30	0.63998658	0.65027209	0.66104996	0.67035969	0.70395857	0.77322723	0.79181516	0.75835687	0.55453757	0.65547032
31	0.88200652	0.88530036	0.88876729	0.89175383	0.90283674	0.9259263	0.93060505	0.93917616	0.96262363	0.92219381

soft-hs

Then we just compare “hard-H” and “hard-hs”; compare “soft-H” and “soft-hs”. In the situation that the value of “hard-H” equal to “hard-hs”, then we choose to hit, otherwise we choose to stand. The same thing will be done to “soft-H” and “soft-hs”.Then we can get the table

hard	2	3	4	5	6	7	8	9	10	ace
4	H	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	H	H
7	H	H	H	H	H	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H
9	H	H	H	H	H	H	H	H	H	H
10	H	H	H	H	H	H	H	H	H	H
11	H	H	H	H	H	H	H	H	H	H
12	H	H	S	S	S	H	H	H	H	H
13	S	S	S	S	S	H	H	H	H	H
14	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	H	H
16	S	S	S	S	S	H	H	H	H	H
17	S	S	S	S	S	S	S	S	S	S
18	S	S	S	S	S	S	S	S	S	S
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

basic strategy considering only standing and hitting-hard

soft	2	3	4	5	6	7	8	9	10	ace
12	H	H	H	H	H	H	H	H	H	H
13	H	H	H	H	H	H	H	H	H	H
14	H	H	H	H	H	H	H	H	H	H
15	H	H	H	H	H	H	H	H	H	H
16	H	H	H	H	H	H	H	H	H	H
17	H	H	H	H	H	H	H	H	H	H
18	S	S	S	S	S	S	S	H	H	H
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

basic strategy considering only standing and hitting-soft

3.2.4 Expected Value when Doubling

We deal with two table named "hard-D" and "soft-D" in this part. As the rule shows, we can know that these two table is just based on "hard-S" and "soft-S" because when doubling, player gets one card only and can not hit again. We get the "hard-D" by applying the following formula

$$P_x^{hard-D}(y) = \begin{cases} \frac{2}{13}(\sum_{n=2}^{9} P_x^{hard-S}(y+n) + 4P_x^{hard-S}(y+10) + P_x^{soft-S}(y+11)), & y = 4 \sim 11 \\ \frac{2}{13}(\sum_{n=1}^{9} P_x^{hard-S}(y+n) + 4P_x^{hard-S}(y+10)), & y = 12 \sim 21 \\ -2, & y = 22 \sim 31. \end{cases} \tag{11}$$

The "soft-D" is given by the formula

$$P_x^{soft-D}(y) = \begin{cases} \frac{2}{13}(\sum_{n=1}^{9} P_x^{soft-S}(y+n) + 4P_x^{soft-S}(y+10)), & y = 12 \sim 21 \\ P_x^{hard-D}(y-10), & y = 22 \sim 31. \end{cases} \tag{12}$$

Then we get the two table

hard-D	2	3	4	5	6	7	8	9	10	ace
4	-0.5855675	-0.5045005	-0.4221262	-0.3343853	-0.307398	-0.9507504	-1.021035	-1.0862994	-1.0808607	-1.3339015
5	-0.5855675	-0.5045005	-0.4221262	-0.3343853	-0.307398	-0.9507504	-1.021035	-1.0862994	-1.0808607	-1.3339015
6	-0.5640584	-0.483726	-0.4020509	-0.3155774	-0.281946	-0.8940479	-1.0012556	-1.0678385	-1.0622899	-1.3048373
7	-0.4357579	-0.3597795	-0.2822991	-0.2027301	-0.1383372	-0.5893359	-0.8470758	-0.9570735	-0.9508656	-1.1304521
8	-0.2044905	-0.1362161	-0.0663721	0.00345644	0.0870152	-0.1877296	-0.4519868	-0.7185013	-0.7465877	-0.810746
9	0.0611185	0.12081635	0.18194893	0.24305722	0.31705475	0.10425035	-0.0264423	-0.3009957	-0.4667067	-0.4329114
10	0.35893941	0.40932067	0.46094024	0.51251711	0.57559017	0.39241246	0.28663572	0.14432837	-0.0086587	-0.0140424
11	0.47064092	0.51779525	0.56604055	0.61469902	0.66738009	0.46288895	0.35069259	0.22778342	0.17968873	0.10906078
12	-0.50678	-0.4673818	-0.4270731	-0.3865423	-0.3410524	-0.5067116	-0.6156609	-0.7375056	-0.7968406	-0.8293439
13	-0.6155825	-0.5824202	-0.548448	-0.5146665	-0.4712526	-0.5874231	-0.6909659	-0.8077903	-0.8675436	-0.8805823
14	-0.724385	-0.6974587	-0.6698229	-0.6427908	-0.6014527	-0.6681346	-0.7662709	-0.8780749	-0.9382466	-0.9318206
15	-0.8331875	-0.8124971	-0.7911978	-0.770915	-0.7316529	-0.7488462	-0.8415759	-0.9483596	-1.0089497	-0.983059
16	-0.94199	-0.9275355	-0.9125727	-0.8990392	-0.861853	-0.8295577	-0.9168809	-1.0186443	-1.0796527	-1.0342973
17	-1.0723016	-1.0633484	-1.054023	-1.0459713	-1.0175052	-0.9669717	-1.0119653	-1.1073898	-1.1689264	-1.1145998
18	-1.2448773	-1.2400099	-1.2349237	-1.2305191	-1.2149581	-1.1822877	-1.1821117	-1.233057	-1.2953416	-1.2530308
19	-1.4581549	-1.4560658	-1.4538743	-1.4519825	-1.4451084	-1.4308995	-1.42732	-1.4311488	-1.4588983	-1.4495901
20	-1.7104605	-1.7099538	-1.7094204	-1.7089609	-1.7072559	-1.7037036	-1.7029838	-1.7016652	-1.6980579	-1.7042779
21	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
22	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
23	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
24	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
25	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
26	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
27	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
28	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
29	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
30	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
31	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2

hard-D

soft-D	2	3	4	5	6	7	8	9	10	ace
12	-0.07157	-0.0072281	0.05842652	0.12595449	0.17974821	-0.1838656	-0.3144409	-0.456367	-0.5140285	-0.6243906
13	-0.07157	-0.0072281	0.05842652	0.12595449	0.17974821	-0.1838656	-0.3144409	-0.456367	-0.5140285	-0.6243906
14	-0.07157	-0.0072281	0.05842652	0.12595449	0.17974821	-0.1838656	-0.3144409	-0.456367	-0.5140285	-0.6243906
15	-0.07157	-0.0072281	0.05842652	0.12595449	0.17974821	-0.1838656	-0.3144409	-0.456367	-0.5140285	-0.6243906
16	-0.07157	-0.0072281	0.05842652	0.12595449	0.17974821	-0.1838656	-0.3144409	-0.456367	-0.5140285	-0.6243906
17	-0.0070427	0.05509528	0.11865255	0.18237816	0.25610429	-0.0137581	-0.2551025	-0.4009845	-0.4583163	-0.537198
18	0.11974956	0.17764128	0.23700385	0.2952255	0.38150648	0.21994797	-0.0299168	-0.2902194	-0.346892	-0.3628128
19	0.24185546	0.29582414	0.35115361	0.40597207	0.47959871	0.31983519	0.19526887	-0.0729455	-0.2354676	-0.1884276
20	0.35893941	0.40932067	0.46094024	0.51251711	0.57559017	0.39241246	0.28663572	0.14432837	-0.0086587	-0.0140424
21	0.47064092	0.51779525	0.56604055	0.61469902	0.66738009	0.46288895	0.35069259	0.22778342	0.17968873	0.10906078
22	-0.50678	-0.4673818	-0.4270731	-0.3865423	-0.3410524	-0.5067116	-0.6156609	-0.7375056	-0.7968406	-0.8293439
23	-0.6155825	-0.5824202	-0.548448	-0.5146665	-0.4712526	-0.5874231	-0.6909659	-0.8077903	-0.8675436	-0.8805823
24	-0.724385	-0.6974587	-0.6698229	-0.6427908	-0.6014527	-0.6681346	-0.7662709	-0.8780749	-0.9382466	-0.9318206
25	-0.8331875	-0.8124971	-0.7911978	-0.770915	-0.7316529	-0.7488462	-0.8415759	-0.9483596	-1.0089497	-0.983059
26	-0.94199	-0.9275355	-0.9125727	-0.8990392	-0.861853	-0.8295577	-0.9168809	-1.0186443	-1.0796527	-1.0342973
27	-1.0723016	-1.0633484	-1.054023	-1.0459713	-1.0175052	-0.9669717	-1.0119653	-1.1073898	-1.1689264	-1.1145998
28	-1.2448773	-1.2400099	-1.2349237	-1.2305191	-1.2149581	-1.1822877	-1.1821117	-1.233057	-1.2953416	-1.2530308
29	-1.4581549	-1.4560658	-1.4538743	-1.4519825	-1.4451084	-1.4308995	-1.42732	-1.4311488	-1.4588983	-1.4495901
30	-1.7104605	-1.7099538	-1.7094204	-1.7089609	-1.7072559	-1.7037036	-1.7029838	-1.7016652	-1.6980579	-1.7042779
31	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2

soft-D

With “hard-D”, “soft-D”, “hard-hs”, “soft-hs”, we can make the two tables “hard-hsd”

and “soft-hsd” which is defined by

$$P_x^{hard-hsd}(y) = max(P_x^{hard-H}(y), P_x^{hard-S}(y), P_x^{hard-D}(y)) \tag{13}$$

$$P_x^{soft-hsd}(y) = max(P_x^{soft-H}(y), P_x^{soft-S}(y), P_x^{soft-D}(y)) \tag{14}$$

hard-hsd	2	3	4	5	6	7	8	9	10	ace
4	−0. 1149133	−0. 0826133	−0. 0493674	−0. 0123799	0. 01113042	−0. 0882792	−0. 1593342	−0. 2406662	−0. 2891979	−0. 253077
5	−0. 1282156	−0. 0953102	−0. 0614795	−0. 023979	−0. 0011863	−0. 1194474	−0. 1880933	−0. 2666151	−0. 3134116	−0. 2785746
6	−0. 1407591	−0. 1072911	−0. 0729171	−0. 034916	−0. 0130058	−0. 1519327	−0. 2172419	−0. 2926407	−0. 3377494	−0. 3041466
7	−0. 1091834	−0. 076583	−0. 0430218	−0. 0072714	0. 02918534	−0. 0688078	−0. 2106048	−0. 2853654	−0. 3190548	−0. 3100717
8	−0. 0217982	0. 00800526	0. 03878447	0. 07080464	0. 11496015	0. 08220744	−0. 0598983	−0. 2101863	−0. 2493751	−0. 1970288
9	0. 07444604	0. 12081635	0. 18194893	0. 24305722	0. 31705475	0. 17186786	0. 09837622	−0. 0521781	−0. 152953	−0. 0656808
10	0. 35893941	0. 40932067	0. 46094024	0. 51251711	0. 57559017	0. 39241246	0. 28663572	0. 14432837	0. 02530852	0. 08144971
11	0. 47064092	0. 51779525	0. 56604055	0. 61469902	0. 66738009	0. 46288895	0. 35069259	0. 22778342	0. 17968873	0. 14300128
12	−0. 25339	−0. 2336909	−0. 2110631	−0. 1671927	−0. 153699	−0. 2128477	−0. 2715748	−0. 3400133	−0. 381043	−0. 3505403
13	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 2690729	−0. 3236052	−0. 3871552	−0. 4252542	−0. 3969303
14	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 321282	−0. 3719191	−0. 4309298	−0. 4663075	−0. 4400067
15	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 3697618	−0. 416782	−0. 4715777	−0. 5044284	−0. 4800062
16	−0. 2927837	−0. 2522502	−0. 2110631	−0. 1671927	−0. 153699	−0. 4147788	−0. 4584404	−0. 5093221	−0. 5398263	−0. 5171487
17	−0. 1529746	−0. 1172162	−0. 0805734	−0. 0449414	0. 01173916	−0. 106809	−0. 381951	−0. 4231542	−0. 4197206	−0. 4780335
18	0. 1217419	0. 14830007	0. 17585444	0. 19956119	0. 28344392	0. 39955417	0. 10595135	−0. 1831634	−0. 1783012	−0. 1001989
19	0. 38630469	0. 40436294	0. 42317892	0. 4395121	0. 49597707	0. 6159765	0. 59385367	0. 28759676	0. 06311817	0. 27763572
20	0. 63998658	0. 65027209	0. 66104996	0. 67035969	0. 70395857	0. 77322723	0. 79181516	0. 75835687	0. 55453757	0. 65547032
21	0. 88200652	0. 88530036	0. 88876729	0. 89175383	0. 90283674	0. 9259263	0. 93060505	0. 93917616	0. 96262363	0. 92219381
22	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
23	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
24	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
25	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
26	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
27	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
28	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
29	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
30	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
31	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1

hard-hsd

soft-hsd	2	3	4	5	6	7	8	9	10	ace
12	0.08183622	0.10350705	0.12659563	0.15648238	0.18595361	0.16547293	0.09511502	6.5791E-05	-0.0700024	-0.0204779
13	0.04663613	0.07411881	0.10247715	0.13336274	0.17974821	0.1223857	0.05405707	-0.0376947	-0.1048514	-0.057308
14	0.02239186	0.05080674	0.08008141	0.12595449	0.17974821	0.07950749	0.01327722	-0.0751632	-0.1394668	-0.0938743
15	-0.0001207	0.02915981	0.05928538	0.12595449	0.17974821	0.03702828	-0.0270548	-0.1121888	-0.1737042	-0.1300265
16	-0.0210252	0.0090591	0.05842652	0.12595449	0.17974821	-0.0048902	-0.0667948	-0.1486435	-0.2074411	-0.1656372
17	-0.000491	0.05509528	0.11865255	0.18237816	0.25610429	0.05382346	-0.0729154	-0.1497869	-0.196867	-0.1795694
18	0.1217419	0.17764128	0.23700385	0.2952255	0.38150648	0.39955417	0.10595135	-0.1007443	-0.1438081	-0.0929355
19	0.38630469	0.40436294	0.42317892	0.4395121	0.49597707	0.6159765	0.59385367	0.28759676	0.06311817	0.27763572
20	0.63998658	0.65027209	0.66104996	0.67035969	0.70395857	0.77322723	0.79181516	0.75835687	0.55453757	0.65547032
21	0.88200652	0.88530036	0.88876729	0.89175383	0.90283674	0.9259263	0.93060505	0.93917616	0.96262363	0.92219381
22	-0.25339	-0.2336909	-0.2110631	-0.1671927	-0.153699	-0.2128477	-0.2715748	-0.3400133	-0.381043	-0.3505403
23	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.2690729	-0.3236052	-0.3871552	-0.4252542	-0.3969303
24	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.321282	-0.3719191	-0.4309298	-0.4663075	-0.4400067
25	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.3697618	-0.416782	-0.4715777	-0.5044284	-0.4800062
26	-0.2927837	-0.2522502	-0.2110631	-0.1671927	-0.153699	-0.4147788	-0.4584404	-0.5093221	-0.5398263	-0.5171487
27	-0.1529746	-0.1172162	-0.0805734	-0.0449414	0.01173916	-0.106809	-0.381951	-0.4231542	-0.4197206	-0.4780335
28	0.1217419	0.14830007	0.17585444	0.19956119	0.28344392	0.39955417	0.10595135	-0.1831634	-0.1783012	-0.1001989
29	0.38630469	0.40436294	0.42317892	0.4395121	0.49597707	0.6159765	0.59385367	0.28759676	0.06311817	0.27763572
30	0.63998658	0.65027209	0.66104996	0.67035969	0.70395857	0.77322723	0.79181516	0.75835687	0.55453757	0.65547032
31	0.88200652	0.88530036	0.88876729	0.89175383	0.90283674	0.9259263	0.93060505	0.93917616	0.96262363	0.92219381

soft-hsd

Compare“hard-hsd” and “soft-hsd” with “hard-S”, “soft-S”, “hard-D”, “soft-D”, we can get

hard	2	3	4	5	6	7	8	9	10	ace
4	H	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	H	H
7	H	H	H	H	H	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H
9	H	D	D	D	D	H	H	H	H	H
10	D	D	D	D	D	D	D	D	H	H
11	D	D	D	D	D	D	D	D	D	H
12	H	H	S	S	S	H	H	H	H	H
13	S	S	S	S	S	H	H	H	H	H
14	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	H	H
16	S	S	S	S	S	H	H	H	H	H
17	S	S	S	S	S	S	S	S	S	S
18	S	S	S	S	S	S	S	S	S	S
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

basic strategy considering only standing, hitting and doubling-hard

soft	2	3	4	5	6	7	8	9	10	ace
12	H	H	H	H	H	H	H	H	H	H
13	H	H	H	H	D	H	H	H	H	H
14	H	H	H	D	D	H	H	H	H	H
15	H	H	H	D	D	H	H	H	H	H
16	H	H	D	D	D	H	H	H	H	H
17	H	D	D	D	D	H	H	H	H	H
18	S	D	D	D	D	S	S	H	H	H
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

basic strategy considering only standing, hitting and doubling-soft

3.2.5 Expected Value when Surrendering

This situation is quite easy, so we will just take a quick look. The "hard-R" and "Soft-R" will be like

hard-R	2	3	4	5	6	7	8	9	10	ace
4	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
6	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
7	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
8	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
9	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
10	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
11	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
12	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
13	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
14	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
15	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
16	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
17	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
18	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
19	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
20	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
21	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5
22	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
23	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
24	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
25	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
26	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
27	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
28	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
29	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
30	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
31	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1

hard-R

soft-R	2	3	4	5	6	7	8	9	10	ace
12	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
13	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
14	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
15	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
16	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
17	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
18	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
19	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
20	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
21	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
22	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
23	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
24	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
25	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
26	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
27	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
28	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
29	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
30	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
31	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5

soft-R

Then we can also get “hard-hsdr” and “soft-hsdr”. Through comparing, we can find that

hard	2	3	4	5	6	7	8	9	10	ace
4	H	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	H	H
7	H	H	H	H	H	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H
9	H	D	D	D	D	H	H	H	H	H
10	D	D	D	D	D	D	D	D	H	H
11	D	D	D	D	D	D	D	D	D	H
12	H	H	S	S	S	H	H	H	H	H
13	S	S	S	S	S	H	H	H	H	H
14	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	R	H
16	S	S	S	S	S	H	H	R	R	R
17	S	S	S	S	S	S	S	S	S	S
18	S	S	S	S	S	S	S	S	S	S
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

basic strategy considering standing, hitting, doubling and surrendering-hard

and the soft one stays the same.

3.2.6 Expected Value when Splitting

To make things easier, we assume that player won't split again after splitting. In this case, we will deal with a table "pair" whose column is consist of 2∼10 and ace and row is pairs of 2∼10 and ace. As for $y = 2 \sim 10$, we can get that

$$P_x^{pair} y = \frac{2}{13}(\sum_{n=2}^{9} P_x^{hard-hsd}(y+n) + 4P_x^{hard-hsd}(y+10) + P_x^{soft-hsd}(y+11)) \tag{15}$$

However, things are kind of different with ace, because player can get only one card after splitting aces. So

$$P_x^{pair}(ace) = \frac{2}{13}(\sum_{y=12}^{20} P_x^{soft-S}(y) + 4P_x^{soft-S}(y)) \tag{16}$$

We get "pair" as below

pair	2	3	4	5	6	7	8	9	10	ace
2	−0. 0888872	−0. 0256161	0. 04294663	0. 12724982	0. 1947786	−0. 0073993	−0. 1741092	−0. 3651212	−0. 4747335	−0. 4067074
3	−0. 1381635	−0. 0638664	0. 01462487	0. 10229275	0. 16942022	−0. 0677605	−0. 2296695	−0. 4151802	−0. 5213959	−0. 455875
4	−0. 1669452	−0. 0913413	−0. 0115874	0. 08025987	0. 14595673	−0. 1294437	−0. 2864541	−0. 4663593	−0. 5691333	−0. 506154
5	−0. 1935497	−0. 1167352	−0. 0329727	0. 05990961	0. 12431163	−0. 1917802	−0. 3439724	−0. 518257	−0. 6175607	−0. 5571492
6	−0. 2186368	−0. 1366784	−0. 0495597	0. 0439869	0. 10792266	−0. 2567507	−0. 4022695	−0. 5703083	−0. 6662363	−0. 6082933
7	−0. 1554854	−0. 0747667	0. 01051147	0. 09996462	0. 18769124	−0. 0905009	−0. 3889953	−0. 5557578	−0. 628847	−0. 6201433
8	0. 0192851	0. 08688786	0. 15656747	0. 2283182	0. 3255334	0. 2115296	−0. 0875823	−0. 4053996	−0. 4894876	−0. 3940576
9	0. 18462902	0. 24214017	0. 30150334	0. 36334825	0. 44337461	0. 37000371	0. 21532327	−0. 0936598	−0. 2966434	−0. 1313616
10	0. 36499999	0. 41217595	0. 46094024	0. 51251711	0. 57559017	0. 51381749	0. 39590742	0. 23305918	0. 05061705	0. 16289942
aces	0. 47064092	0. 51779525	0. 56604055	0. 61469902	0. 66738009	0. 46288895	0. 35069259	0. 22778342	0. 17968873	0. 10906078

pair

Then we can determine whether to split or not by compare the expected value of splitting the pair of x and hard $2x$. So we have the table "best" defined by

$$P_x^{best}(y) = \begin{cases} max(P_x^{pair}(y), P_x^{hard-hsd}(2y)), & y = 2 \sim 10 \\ max(P_x^{pair}(y), P_x^{soft-hsd}(12)), & y = ace. \end{cases} \tag{17}$$

best	2	3	4	5	6	7	8	9	10	ace
2	−0. 0888872	−0. 0256161	0. 04294663	0. 12724982	0. 1947786	−0. 0073993	−0. 1593342	−0. 2406662	−0. 2891979	−0. 253077
3	−0. 1381635	−0. 0638664	0. 01462487	0. 10229275	0. 16942022	−0. 0677605	−0. 2172419	−0. 2926407	−0. 3377494	−0. 3041466
4	−0. 0217982	0. 00800526	0. 03878447	0. 08025987	0. 14595673	0. 08220744	−0. 0598983	−0. 2101863	−0. 2493751	−0. 1970288
5	0. 35893941	0. 40932067	0. 46094024	0. 51251711	0. 57559017	0. 39241246	0. 28663572	0. 14432837	0. 02530852	0. 08144971
6	−0. 2186368	−0. 1366784	−0. 0495597	0. 0439869	0. 10792266	−0. 2128477	−0. 2715748	−0. 3400133	−0. 381043	−0. 3505403
7	−0. 1554854	−0. 0747667	0. 01051147	0. 09996462	0. 18769124	−0. 0905009	−0. 3719191	−0. 4309298	−0. 4663075	−0. 4400067
8	0. 0192851	0. 08688786	0. 15656747	0. 2283182	0. 3255334	0. 2115296	−0. 0875823	−0. 4053996	−0. 4894876	−0. 3940576
9	0. 18462902	0. 24214017	0. 30150334	0. 36334825	0. 44337461	0. 39955417	0. 21532327	−0. 0936598	−0. 1783012	−0. 1001989
10	0. 63998658	0. 65027209	0. 66104996	0. 67035969	0. 70395857	0. 77322723	0. 79181516	0. 75835687	0. 55453757	0. 65547032
aces	0. 47064092	0. 51779525	0. 56604055	0. 61469902	0. 66738009	0. 46288895	0. 35069259	0. 22778342	0. 17968873	0. 10906078

best

Compare the “pair” and “best”, we can get “split” in which “Y” shows that you should split and “N” not.

split	2	3	4	5	6	7	8	9	10	ace
2	Y	Y	Y	Y	Y	Y	N	N	N	N
3	Y	Y	Y	Y	Y	Y	N	N	N	N
4	N	N	N	Y	Y	N	N	N	N	N
5	N	N	N	N	N	N	N	N	N	N
6	Y	Y	Y	Y	Y	N	N	N	N	N
7	Y	Y	Y	Y	Y	Y	N	N	N	N
8	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
9	Y	Y	Y	Y	Y	N	Y	Y	N	N
10	N	N	N	N	N	N	N	N	N	N
aces	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

split

4 Conclusion

After all these analysis, we can clearly learn how the basic strategy for blackjack is come up with and everyone can draw these three tables by excel within a short time. It can also help us to really understand the principle of mathematics behind the blackjack.

5 Reference

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